Class : XII

MATHEMATICS

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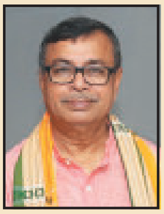
রতন লাল নাথ
মন্ত্রী
শিক্ষা দপ্তর
ত্রিপুরা সরকার

বার্তা



শিক্ষার প্রকৃত বিকাশের জন্য, শিক্ষাকে যুগোপযোগী করে তোলার জন্য প্রয়োজন শিক্ষাসংক্রান্ত নিরন্তর গবেষণা। প্রয়োজন শিক্ষা সংশ্লিষ্ট সকলকে সময়ের সঙ্গে সঙ্গে প্রশিক্ষিত করা এবং প্রয়োজনীয় শিখন সামগ্রী, পাঠ্যক্রম ও পাঠ্যপুস্তকের বিকাশ সাধন করা। এস সি ই আর টি ত্রিপুরা রাজ্যের শিক্ষার বিকাশে এসব কাজ সুনামের সঙ্গে করে আসছে। শিক্ষার্থীর মানসিক, বৌদ্ধিক ও সামাজিক বিকাশের জন্য এস সি ই আর টি পাঠ্যক্রমকে আরো বিজ্ঞানসম্মত, নান্দনিক এবং কার্যকর করবার কাজ করে চলেছে। করা হচ্ছে সুনির্দিষ্ট পরিকল্পনার অধীনে।

এই পরিকল্পনার আওতায় পাঠ্যক্রম ও পাঠ্যপুস্তকের পাশাপাশি শিশুদের শিখন সক্ষমতা বৃদ্ধির জন্য তৈরি করা হয়েছে ওয়ার্ক বুক বা অনুশীলন পুস্তক। প্রসঙ্গত উল্লেখ্য, ছাত্র-ছাত্রীদের সমস্যার সমাধানকে সহজতর করার লক্ষ্যে এবং তাদের শিখনকে আরো সহজ ও সাবলীল করার জন্য রাজ্য সরকার একটি উদ্যোগ গ্রহণ করেছে, যার নাম ‘প্রয়াস’। এই প্রকল্পের অধীনে এস সি ই আর টি এবং জেলা শিক্ষা আধিকারিকরা বিশিষ্ট শিক্ষকদের সহায়তা গ্রহণের মাধ্যমে প্রথম থেকে দ্বাদশ শ্রেণির ছাত্র-ছাত্রীদের জন্য ওয়ার্ক বুকগুলো সুচারুভাবে তৈরি করেছেন। ষষ্ঠ থেকে অষ্টম শ্রেণি পর্যন্ত বিজ্ঞান, গণিত, ইংরেজি, বাংলা ও সমাজবিদ্যার ওয়ার্ক বুক তৈরি হয়েছে। নবম দশম শ্রেণির জন্য হয়েছে গণিত, বিজ্ঞান, সমাজবিদ্যা, ইংরেজি ও বাংলা। একাদশ দ্বাদশ শ্রেণির ছাত্র-ছাত্রীদের জন্য ইংরেজি, বাংলা, হিসাবশাস্ত্র, পদার্থবিদ্যা, রসায়নবিদ্যা, অর্থনীতি এবং গণিত ইত্যাদি বিষয়ের জন্য তৈরি হয়েছে ওয়ার্ক বুক। এইসব ওয়ার্ক বুকের সাহায্যে ছাত্র-ছাত্রীরা জ্ঞানমূলক বিভিন্ন কার্য সম্পাদন করতে পারবে এবং তাদের চিন্তা প্রক্রিয়ার যে স্বাভাবিক ছন্দ রয়েছে, তাকে ব্যবহার করে বিভিন্ন সমস্যার সমাধান করতে পারবে। বাংলা ও ইংরেজি উভয় ভাষায় লিখিত এইসব অনুশীলন পুস্তক ছাত্র-ছাত্রীদের মধ্যে বিনামূল্যে বিতরণ করা হবে।

এই উদ্যোগে সকল শিক্ষার্থী অতিশয় উপকৃত হবে। আমার বিশ্বাস, আমাদের সকলের সক্রিয় এবং নিরলস অংশগ্রহণের মাধ্যমে ত্রিপুরার শিক্ষাজগতে একটি নতুন দিগন্তের উন্মেষ ঘটবে। ব্যক্তিগত ভাবে আমি চাই যথাযথ জ্ঞানের সঙ্গে সঙ্গে শিক্ষার্থীর সামগ্রিক বিকাশ ঘটুক এবং তার আলো রাজ্যের প্রতিটি কোণে ছড়িয়ে পড়ুক।

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CONTENTS

CHAPTER- 1

RELATIONS AND FUNCTIONS

Important points and Results :

- **Relation :**
- A relation from a set A to a set B is a subset of A×B.
- Total number of relations from a set consisting of p elements to a set consisting of q elements is 2^{pq}.
- A relation on a set A is a subset of A×A.
- A relation R on a set A is said to be
  i) reflexive, if $(a,a)\in R$, for all $a\in A$.
  ii) symmetric, if $(a,b)\in R \Rightarrow (b,a)\in R$ for all $a,b\in A$
  iii) transitive, if $(a,b)\in R$ and $(b,c)\in R \Rightarrow (a,c)\in R$, for all $a,b,c\in R$
  iv) anti symmetric, if $(a,b)\in R$ and $(b,a)\in R \Rightarrow a=b$
  v) an equivalence relation, if it is reflexive, symmetric and transitive.
- **Some results :**
  i) The intersection of two equivalence relation on a set A is an equivalence relation on set A.
  ii) The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
  iii) Inverse of an equivalence relation is an equivalence relation.
  iv) The identity relation on a non-empty set is always equivalence relation.
  v) The identity relation on a non-empty set is always an anti symmetric relation.
  vi) The universal relation on a non-empty set is always an equivalence relation.
  vii) If X, Y, Z be three non-emtpy sets and $R \subseteq X\times Y$, $S \subseteq Y\times Z$, then compositon of relation R and S deonoted by $SoR \subseteq X\times Z$.
  viii) If R be a relation from a set of X to Y and S be a relation from Y to Z then $(RoS)^{-1}=S^{-1}oR^{-1}$
  ix) Number of relations from a set A to A having n elements are equal to $2^{nxn}=2^{n^2}$
  x) Let A and B be two non-empty sets having n elements in common then the number of elements common in $(A\times B)\cap(B\times A)$ are $= n\times n=n^2$.
- **Function :**

  Let A and B be two non-empty sets. Then, a subset f of A×B is a function from A to B, if
  i) for each $a\in A$ there exists $b\in B$ such that $(a,b)\in f$
  ii) $(a,b)\in f$ and $(a,c)\in f \Rightarrow b=c$
- If f is a function from set A to set B, we write it as $f: A\rightarrow B$.

  The set A is called the domain of f and the set B is known as its co-domain. The set of images of elements of set A is known as the range of f.
- If $f: A\rightarrow B$ is a funciton, then $x = y \Rightarrow f(x)=f(y)$ for all $x, y\in A$.

- **Remark :**

i) We denote the function by $f, g, h, \phi, \psi, \zeta$ etc.

ii) A function is also named as 'mapping'.

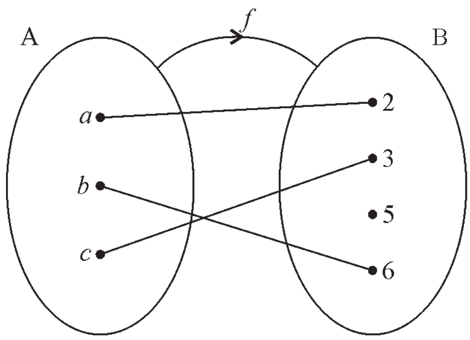
iii) A function is a particular type of relation.

- **Some special type of function :**

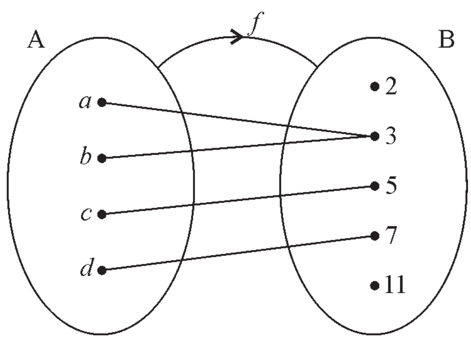
i) One-one function or injective function :

A mapping f from A to B is said to be one-one if every element in A there exists a unique element in set B.

In other words, no element of B is the image of more than one element of A. i.e., image of distinct elements of A are also distinct element in set B.



one-one



Not one-one

Mathematically, a function f : A→B is a one-one or injective function if, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in$ A.

OR

$x \neq y \Rightarrow f(x) \neq f(y)$, for all $x, y \in$ A.

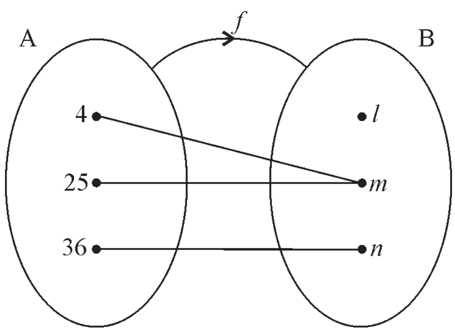
Note : A function is one-one, if it is either strictly increasing or strictly decreasing.

ii) Many-one function :

A function f : A→B is said to be many-one function if two or more elements of set A have the same image in B.

Thus, f : A→B is a many-one function if there exist $x, y \in$ A such that $x \neq y$ but $f(x) = f(y)$.

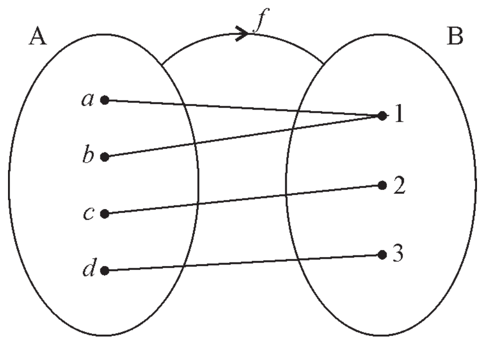
In other words, f : A→B is a many-one function if it is not a one-one function.



iii) Onto or surjective function :

A function $f: A \to B$ is said to be onto or surjective if f for each $b \in B$, there exists $a \in A$ such that $f(a)=b$.

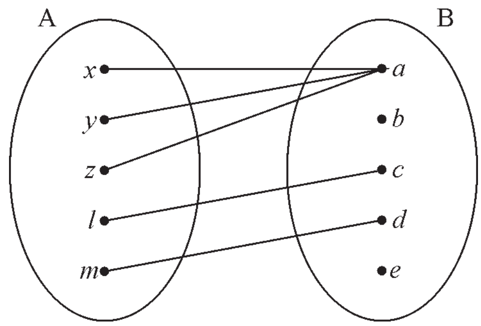
In other words, a function $f: A \to B$ is surjective if range (f) = co-domain (f)



iv) Into function :

A function $f: A \to B$ is an into function if there exists an element in B having no pre-image in A i.e., Range $(f) \subset$ co-domain (f).

In other words, $f: A \to B$ is an into function if it is not an onto function.



v) Bijective (one-one and onto) function :

A function $f: A \to B$ is bijective if it is one-one and onto.

In other words, a function $f: A \to B$ is bijective, if

a) $f(x) = f(y) \Rightarrow x = y \ \forall x, y \in A$

b) for all $y \in B$, there exists $x \in A$ such that $f(x)=y$.

Note : If a function $f: A \to A$ is one-one then it is obviously onto.

- If a function $f: A \to B$ is not an onto function, then $f: A \to f(A)$ is always an onto function.
- Let A and B be two finite sets and $f: A \to B$ be a function.

i) If f is injective, then $n(A) \leq n(B)$

ii) if f is surjective, then $n(A) \geq n(B)$

iii) If f is bijective, then $n(A) = n(B)$

- If A and B are two non-empty finite sets such that $n(A) = m$ and $n(B) = n$, then

i) number of functions from A to B = n^m

ii) number of one-one functions from A to B = $\begin{cases} {}^n c_m \times m!, \ i.e\ {}^n P_m & if\ \ n \ge m \\ 0, & if\ \ n < m \end{cases}$

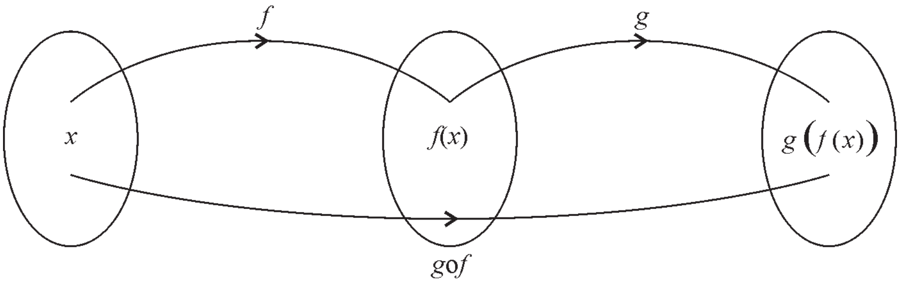
iii) number of on to functions from A to B = $\begin{cases} \sum_{r=1}^{n} (-1)^{n-r}\ {}^n c_r\, r^m, & if\ \ m \ge n \\ 0, & if\ \ m < n \end{cases}$

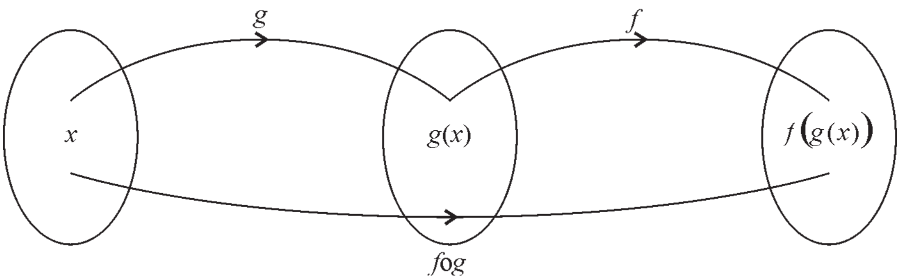
iv) number of one-one and onto functions from A to B = $\begin{cases} n!, & if\ \ m = n \\ 0, & if\ \ m \ne n \end{cases}$

- **Composite function or function of a function :**

Let f : A→B and g : B→C be two functions. Then, a function gof : A→C defined by $(gof)(x) = g\left(f(x)\right)$, for all $x \in$ A is called the composition of f and g.

Note : For the composition gof to exist, the range of f must be a subset of the domain of g. Similarly, fog exist if range of g is a subset of domain of f.





Some results :

i) $fog \ne gof$

ii) $(fog)oh = fo(gof)$

iii) the composition of two bijections is a bijection.

iv) f : A→B. Then, $foI_A = I_B of = f$.

v) Let f : A→B, g : B→A be two functions such that $gof = I_A$. Then f is an injection and g is a surjection. In other words, if $fog = I_B$, then f is a surjection and g is an injection.

vi) Let $f : A \to B$ and $g : B \to C$ be two functions. Then,

a) $gof : A \to C$ is onto $\Rightarrow g : B \to C$ is onto.

b) $gof : A \to C$ is one-one $\Rightarrow f : A \to B$ is one-one.

c) $gof : A \to C$ is onto and $g : B \to C$ is one-one $\Rightarrow f : A \to B$ is onto.

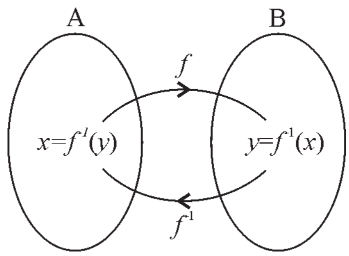
d) $gof : A \to C$ is one-one and $f : A \to B$ is onto $\Rightarrow g : B \to C$ is one-one.

- **Inverse of a function :**

Let $f : A \to B$ be a bijection. Then a function $g : B \to A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of f.

i.e., $f(x) = y \Leftrightarrow g(y) = x$.

The inverse of f is generally denoted by f^{-1}. Thus, if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ such that $f(x) = y \Leftrightarrow f^{-1}(y) = x$.



- **Some results :**

i) The inverse of a bijection is unique.

ii) The inverse of a bijection is also a bijection.

iii) If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity functions on the sets A and B respectively.

iv) If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is a bijection and $(gof)^{-1} = f^{-1}og^{-1}$.

v) Let $f : A \to B$ and $g : B \to A$ such that $gof = I_A$ and $fog = I_B$. Then, f and g are bijections and $g = f^{-1}$.

vi) Let $f : A \to B$ be an invertible function. Then $\left(f^{-1}\right)^{-1} = f$.

- **Binary operations :**

A binary operation on a set S is a function form $S \times S$ to S i.e., $f : S \times S \to S$. A binary operation $*$ on a set S associates any two elements $a, b \in S$ to a unique element $a * b \in S$.

- A binary operation $*$ on a set S is said to be

i) Commutative, if $a * b = b * a$, $\forall a, b \in S$.

ii) Associative, if $(a * b) * c = a * (b * c)$, $\forall a, b, c \in S$.

iii) Distributive over a binary operation 'o' on S, if $a * (boc) = (a * b)o(a * c)$ and, $(boc) * a = (b * a)o(c * a)$, $\forall a, b, c \in S$.

- An element $e \in S$ is said to be identity element for the binary operation $*$, if $a * e = a = e * a,\ \forall a \in S$.
- Let $*$ be a binary operation on a set S and $e \in S$ be the identity element. An element $a \in S$ is said to be invertible, if there exists an element $b \in S$ such that $a * b = e = b * a$

- **Composition Table :**

  A binary operation on a finite set can be completely described by means of a table known as a compositon table.

  Let $S = \{a_1, a_2, \ldots\ldots\ldots., a_n\}$ be a finite set and $*$ be a binary operation of S. Then the compositon table for $*$ is given by

$*$	a_1	a_2	a_3	....	a_n
a_1	$a_1 * a_1$	$a_1 * a_2$	$a_1 * a_3$	....	$a_1 * a_n$
a_2	$a_2 * a_1$	$a_2 * a_2$	$a_2 * a_3$	....	$a_2 * a_n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_n	$a_n * a_1$	$a_n * a_2$	$a_n * a_3$	....	$a_n * a_n$

From the composition table we infer the following properties :

i) $*$ is commulative if the compositon table is symmetric about the leading diagonal.

ii) If the row headed by an element say e coincides with row at the top and the column headed by e coincides with the coloumn on the extreme left, then e is the identity element.

iii) If each row, except the top-most row, or each column, except the left-most column, contains the identity element. Then, every element of the set is invertible with respect to the given binary operation.

- Total number of binary operations on a set consisting n elements is n^{n^2}.
- Total number of commutative binary operations on a set consisting of n elements is $n^{\frac{n(n-1)}{2}}$.
- **Addition modulo *n* :**

  Let n be a positive integer greater than 1 and $a, b \in Z_n$, where $Z_n = \{0, 1, 2, \ldots\ldots, (n-1)\}$. Then, we define addition modulo n i.e, t_n as follows –

  $a +_n b$ = least non-negative remainder when $a+b$ is divided by n.

- **Multiplication modulo *n* :**

  Let n be a positive integer greater than 1 and $a, b \in Z_n$, where $Z_n = \{0, 1, 2, \ldots.(n-1)\}$.

  Then, we define multiplication modulo n i.e. $\times_n$ as follows –

  $a \times_n b$ = least non-negative remainder when ab is divided by n.

Excercise-1

Section-A

OBJECTIVE TYPE QUESTIONS : [Each question carries 1 or 2 marks]

1. Multiple choice type questions :

i) R is a relation on the set A= $\{1,2,3,4\}$ given by R=$\{(1,2),(2,2),(1,1),(4,4),(1,3),(3,3),(3,2)\}$. Then
a) R is reflexive and symmetric but not transitive.
b) R is reflexive and transitive but not symmetric.
c) R is symmetric and transitive but not reflexive.
d) R is an equivalence relation.

ii) A relation ψ from **C** to **R** is defined by $x\psi y \Leftrightarrow |x| = y$. Which one is correct ?
a) $(2+3i)\ \psi\ 13$ b) $3\ \psi\ (-3)$ c) $(1+i)\ \psi\ 2$ d) $i\ \psi\ 1$

iii) R be a relation over the set of all straight lines in a plane such that $l_1 R l_2 \Leftrightarrow l_1 \perp l_2$. Then R is
a) symmetric b) reflexive c) transitive d) an equivalence relation

iv) If R is the largest equivalence relation on a set A and S is any relation on A, then
a) $R \subset S$ b) $S \subset R$ c) R=S d) none of these.

v) If R is a relation on the set A =$\{1,2,3\}$ given by R =$\{(1,1),(2,2),(3,3)\}$, then R is
a) reflexive b) symmetric c) transitive d) all the three options.

vi) The relation 'R' in N×N such that $(a,b)R(c,d) \Leftrightarrow a+d=b+c$ is
a) reflexive but not symmetric b) reflexive and transitive but not symmetric
c) an equivalence relation d) none of these.

vii) Which of the following is not an equivalence relation on Z ?
a) $aRb \Leftrightarrow a+b$ is an even integer b) $aRb \Leftrightarrow a-b$ is an even integer
c) $aRb \Leftrightarrow a<b$ d) $aRb \Leftrightarrow a=b$

viii) S is a relation over the set R of all real numbers and it is given by $(a,b) \in S \Leftrightarrow ab \geq 0$. Then, S is
a) symmetric and transitive only b) reflexive and symmetric only
c) a partial order relation d) an equivalence relation

ix) R is a relation on the set Z of integers and it is given by $(x,y) \in R \Leftrightarrow |x-y| \leq 1$. Then, R is
a) reflexive and transitive b) reflexive and symmetric
c) symmetric and transitive d) an equivalence relation

x) f: R→R given by $x+\sqrt{x^2}$ is
a) injective b) surjective c) bijective d) none of these.

xi) The function f: R→R defined by $f(x) = 2^x + 2^{|x|}$ is
a) one-one and onto b) many-one and onto
c) one-one and into d) many-one and into

xii) The function f: A→B defined by $f(x) = -x^2+6x-8$ is a bijection, if
a) $A = (-\infty, 3]$ and $B = (-\infty, 1]$ b) $A = [-3, \infty)$ and $B = (-\infty, 1]$

c) $A=(-\infty,3]$ and $B=[1,\infty)$ d) $A=[3,\infty)$ and $B=[1,\infty)$

xiii) Let f : R→R be given by $f(x) = [x]^2+[x+1]-3$, where $[x]$ denotes the greatest integer less than or equal to x. Then, $f(x)$ is

a) many-one and onto b) many-one and into
c) one-one and into d) one-one and onto

xiv) A function f from the set of natural numbers to integers defined by

$$f(n)=\begin{cases}\dfrac{n-1}{2}, & when\ n\ is\ odd\\ -\dfrac{n}{2}, & when\ n\ is\ even\end{cases}$$

a) neither one-one nor onto b) one-one but not onto
c) onto but not one-one d) one-one and onto both

xv) Let $f(x)=x^2$ and $g(x)=2^x$. Then the solution set of the equatoin $fog(\mathrm{x})=gof(x)$ is
a) R b) $\{0\}$ c) $\{0,2\}$ d) none of these

xvi) The inverse of the function f: R→$\{x\in\mathrm{R} : x<1\}$ given by $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ is

a) $\frac{1}{2}\log\frac{1+x}{1-x}$ b) $\frac{1}{2}\log\frac{2+x}{2-x}$ c) $\frac{1}{2}\log\frac{1-x}{1+x}$ d) none of these

xvii) Let $f(x)=\dfrac{1}{1-x}$. Then $\{fo(fof)\}(x)$

a) $x,\ \forall x\in\mathrm{R}$ b) $x,\ \forall x\in\mathrm{R}-\{1\}$ c) $x,\ \forall x\in\mathrm{R}-\{0,1\}$ d) none of these

xviii) Let $f(x)=\dfrac{\alpha x}{x+1}, x\neq-1$. Then, for what value of α is $f(f(x))=x$?

a) $\sqrt{2}$ b) $-\sqrt{2}$ c) 1 d) –1

xix) Let $f(\mathrm{x})=x^3$ be a function with domain $\{0,1,2,3\}$. Then domain of f^{-1} is

a) $\{3,2,1,0\}$ b) $\{0,-1,-2,-3\}$ c) $\{0,1,8,27\}$ d) $\{0,-1,-8,-27\}$

xx) If $g(x) = x^2+x-2$ and $\frac{1}{2}$ $gof(x)=2x^2-5x+2$, then $f(x)$ is equal to

a) $2x-3$ b) $2x+3$ c) $2x^2+3x+1$ d) $2x^2-3x-1$

xxi) If $f(x) = \sin^2 x$ and $g(f(x))=|\sin x|$, then $g(x)$ is equal to

a) $\sqrt{x-1}$ b) $\sqrt{x}$ c) $\sqrt{x+1}$ d) $-\sqrt{x}$

xxii) If f : R→R is given by $f(x)=x^3+3$, then $f^{-1}(x)$ is equal to

a) $x^{1/3}-3$ b) $x^{1/3}+3$ c) $(x-3)^{1/3}$ d) $x+3^{1/3}$

xxiii) If $a*b=a^2+b^2$, then the value of $(4*5)*3$ is
a) $(4^2+5^2)+3^2$ b) $(4+5)^2+3^2$ c) 41^2+3^2 d) $(4+5+3)^2$

xxiv) Q^+ denote the set of all positive rational numbers. If the binary operation $\odot$ on Q^+ is defined as $a \odot b = \frac{ab}{2}$, then the inverse of 3 is

a) $\frac{4}{3}$ b) 2 c) $\frac{1}{3}$ d) $\frac{2}{3}$

xxv) Let $*$ be a binary operation defined on set $Q - \{1\}$ by the rule $a*b = a+b-ab$. Then, the identity element for $*$ is

a) 1 b) $\frac{a-1}{a}$ c) $\frac{a}{a-1}$ d) 0

xxvi) On Z an operation $*$ is defined by $a*b = a^2+b^2$ for all $a,b \in Z$. Then, the operation $*$ on Z is

a) commutative and associative b) associative but not commutative
c) not associative d) not a binary operation

xxvii) Let * be a binary operation on Q^+ defined by $a*b = \frac{ab}{100}$ for all $a,b \in Q^+$. The inverse of 0.1 is

a) 10^5 b) 10^4 c) 10^6 d) none of these

xxviii) If G is the set of all matrices of the form $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$, where $x \in \mathbf{R} - \{0\}$, then the identity element with respect to the multiplication of matrices as binary operation, is

a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ c) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ d) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

xxix) The number of binary operations that can be defined on a set of 2 elements is

a) 8 b) 4 c) 16 d) 64

xxx) Let $A = \{a,b,c,d,e\}$, $B = \{p,q\}$, then the number of onto function from A to B are

a) 8 b) 6 c) 30 d) 32

2. **Very short answer type questions :**

i) Check whether the relation R defined on the set of real numbers such that $R = \{(a,b) : a < b\}$ is transitive or not.

ii) If $f : A \rightarrow B$ is bijective function such that n (A) = 10, then n(B) = ?

iii) Check whether the function $f : R \rightarrow R$ defined as $f(x) = x^3$ is one-one or not.

iv) How many reflexive relations are possible in set A whose $n(A)=3$?

v) A relation R on $S = \{1,2,3\}$ is defined as $R = \{(1,1),(1,2),(2,2),(3,3)\}$. Which element(s) of relation R be removed to make R an equivalence relation ?

vi) A relation R in the set of real numbers defined as $R=\{(a,b):\sqrt{a}=b\}$ is a function or not. Justify.

vii) An equivalence relation R in X divides it into equivalence classes X_1, X_2, X_3. What is the value of $X_1 \cup X_2 \cup X_3$ and $X_1 \cap X_2 \cap X_3$?

viii) If $f(x) = x+7$ and $g(x) = x-7$, $x \in R$, then find $(fog)(x)$.

ix) Is it true that every relation which is symmetric and transitive is also reflexive ? Give reasons.

x) Check whether the relation R defined on the set of real numbers such that $R=\{(a,b):a \le b^3\}$ is transitive or not.

xi) Give an example of a function which is one-one but not onto.

xii) If n(A) = 5, then write the number of one-one function from A to A.

xiii) If A = { 1,2,3 } and B = { a,b }. Write total number of functions from A to B.

xiv) Is the relation R in the set A = { 1,2,3,4,5 } defined as $R=\{(a,b):b=a+1\}$ reflexive ?

xv) If n(A)=n(B)=3, then how many bijecitve functions from A to B can be formed ?

Section-B

3. Short answer type questions : [3 Marks for each question]

i) If $f:\{1,3\}\to\{1,2,5\}$ and $g:\{1,2,5\}\to\{1,2,3,4\}$ be given by $f=\{(1,2),(3,5)\}$, $g=\{(1,3),(2,3),(5,1)\}$, write down gof.

ii) If f: R→R defined by $f(x)=\dfrac{2x-1}{5}$ be an invertible function, write $f^{-1}(x)$.

iii) Let $*$ be a binary operation defined on R, then if $a*b=\dfrac{(a+b)^2}{3}$, write $(2*3)*4$.

iv) If $f: R \to A$, given by $f(x) = x^2 - 2x + 2$ is onto function, then find the set A.

v) If $f(n)=\dfrac{4x+3}{6x-4}$, $x \ne \dfrac{2}{3}$, then find f^{-1}.

vi) If $f: R \to R$ and $g: R \to R$ and defined by $f(x) = 2x + x^2$ and $g(x) = x^3$. Than find fog.

vii) If $f(x)=\log\left(\dfrac{1+x}{1-x}\right)$, show that $f\left(\dfrac{2x}{1+x^2}\right)=2f(x)$.

viii) If $*$ is a binary operation on N defined by $a*b=a+ab$, $\forall a,b \in N$, write the identity element in N if it exists.

ix) If f: R→R, given by $f(x) = |x-1|$ is one-one ?

x) Given a relation R = { (7,8), (8,3) } on the set A = { 3,7,8 }. What is the least number of ordered pairs which when added to R to make it an equivalence relation ? Give reason.

xi) Find the domain of the function.

$$f(x)=\log_4\left(\log_5\left(\log_3\left(18x-x^2-77\right)\right)\right)$$

xii) Find the inverse of the function $f(x)=\log_2\left(x+\sqrt{x^2+1}\right)$

xiii) Let f : R→R, g : R→R be two functions given by $f(x) = 5x-4$ and $g(x)=x^3+7$ then $(fog)^{-1}(x) = ?$

xiv) The set $S = \{1, 2, 3, ..., 12\}$ is to be partitioned into three sets A,B,C of equal size. Thus $A \cup B \cup C = S$, $A \cap B = B \cap C = C \cap A = \phi$. The number of ways to partition S = ?

xv) For real x, check whether the function $f(x) = x^3+5x+1$ is bijective or not. Give reason.

Section-C

4] Long answer type questions : [each questions carries 4 or 6 marks]

i) Let N denote the set of all natural numbers and R be the relation on N×N defined by $(a,b)R(c,d) \Leftrightarrow ad(b+c) = bc(a+d)$. Check whether R is an equivalence relation on N×N or not added.

ii) Let f : N→N be defined by $f(n) = \begin{cases} n+1, & if\ n\ is\ odd \\ n-1, & if\ n\ is\ even \end{cases}$.

Show that f is a bijection.

iii) Show that f : R→R, given by $f(x) = x-[x]$, is neither one-one nor onto.

iv) Let $A = \{x \in R : 0 \le x \le 1\}$. If f : A→A is defined by $f(x) = \begin{cases} x, & if\ x \in Q \\ 1-x, & if\ x \notin Q \end{cases}$, then prove that $fof(x) = x,\ \forall\ x \in A$.

v) If $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ and $g : [-1,1] \to R$ be defined as $f(x) = \tan x$ and $g(x) = \sqrt{1-x^2}$ respectively. Describe fog and gof.

vi) If $f(x) = \left(a - x^n\right)^{\frac{1}{n}}$, where $a>0$ and $n \in N$. Show that $f(f(x)) = x$.

vii) If f : Q→Q, g : Q→Q are two functions defined by $f(x) = 2x$ and $g(x) = x+2$. Show that f and g are bijective maps. Also, verify that $(gof)^{-1} = f^{-1}og^{-1}$

viii) Write the multiplication table for the set of integers modulo 5.

ix) Construct the compostion table for $+_5$ on the set S = { 0,1,2,3,4 }

x) Let $+_6$(addition modulo 6) be a binary operation on S = { 0,1,2,3,4,5 }. Write the value of $2 +_6 4^{-1} +_6 3^{-1}$.

xi) Check the following functions for one-one and onto

a) f : R→R, $f(x) = \frac{2x-3}{7}$ b) f : R→R, $f(x) = |x+1|$

c) f : R–{ 2 }→R, $f(x) = \frac{3x-1}{x-2}$ d) f : R→[–1,1], $f(x) = \sin^2 x$

xii) f : R→R, g : R→R given by $f(x) = [x]$, $g(x) = |x|$ then find $(fog)\left(-\frac{2}{3}\right)$ and $(gof)\left(-\frac{2}{3}\right)$.

xiii) If $*$ is a binary operation defined on R–{ 0 } defined by $a*b = \frac{2a}{b^2}$, then check $*$ for commutativity and associativity.

ANSWERS

Section-A

A].1. i) b ii) d iii) a iv) b v) d vi) c vii) c
viii) d ix) b x) d xi) c xii) a xiii) b xiv) d xv) c
xvi) a xvii) c xviii) d xix) c xx) a xxi) b xxii) c
xxiii) c xxiv) a xxv) d xxvi) c xxvii) a xxviii) c xxix) c xxx) c

2. i) R is transitive ii) 10 iii) $f(x)$ is one-one iv) 2^6

v) (1,2) vi) It is not a function as for $a \in (-\alpha, 0)$, $\sqrt{a}$ is not defined

vii) $X_1 \cup X_2 \cup X_3 = X$ and $X_1 \cap X_2 \cap X_3 = \phi$ viii) x ix) No x) not transitive
xii) 120 xiii) 8 xiv) No xv) 6

Section-B

3. i) $\{(1,3),(3,1)\}$ ii) $f^{-1}(x) = \dfrac{5x+1}{2}$ iii) $\dfrac{1369}{27}$ iv) $A=[1,\infty)$

v) $f^{-1} = \dfrac{3+4y}{6y-4}$ vi) $2x^3 + x^6$ vii) Does not exist

ix) f is not one-one. since, $f(3) = f(-1) = 2$ x) 7 xi) $x \in (8,10)$

xii) $f^{-1}(x) = \dfrac{1}{2}\left(2^x - 2^{-x}\right)$ xiii) $\left(\dfrac{x-31}{5}\right)^{\frac{1}{3}}$ xiv) $\dfrac{12!}{(4!)^3}$ xv) f is bijective.

Section-C

4. i) Yes, R is an equivalence relation v) $fog : [-1,1] \to R$ is defined as $fog(x) = \tan\sqrt{1-x^2}$

$gof : \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right] \to R$ is defined as $gof(x) = \sqrt{1-\tan^2 x}$

viii)

X_5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

ix)

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

x) 1

xi) (a) Bijective (b) Neither one-one nor onto (c) one-one, but not onto (d) Neither one-one nor onto

xii) $(fog)\left(-\dfrac{2}{3}\right) = 0$ *and* $(gof)\left(-\dfrac{2}{3}\right) = 1$ xiii) Neither Commutative nor associative

CHAPTER- 2

INVERSE TRIGONOMETRIC FUNCTION

Important points and Results :

- The functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\text{cosec}^{-1}x$ are called inverse trionometric function.
- Note that $\sin^{-1}x \neq \frac{1}{\sin x}$ and $\left(\sin^{-1}x\right)^2 \neq \sin^{-2}x$. Also $\sin^{-1}x \neq (\sin x)^{-1}$.
- Table for domain, range and principal value ($n \in Z$) of trigonometric inverse functions :

Function	Domain	Range	Principal value
$\sin^{-1}x$	$[-1,1]$	$n\pi+(-1)^n\alpha$	α, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$
$\cos^{-1}x$	$[-1,1]$	$2n\pi \pm \alpha$	α, where $0 \leq \alpha \leq \pi$
$\tan^{-1}x$	$(-\alpha, \alpha)$	$n\pi+\alpha$	α, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$
$\cot^{-1}x$	$(-\alpha, \alpha)$	$n\pi+\alpha$	α, where $0 < \alpha < \pi$
$\text{cosec}^{-1}x$	$\|x\| \geq 1$	$n\pi+(-1)^n\alpha$	α, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, $\alpha \neq 0$
$\sec^{-1}x$	$\|x\| \geq 1$	$2n\pi \pm \alpha$	α, where $0 \leq \alpha \leq \pi$, $\alpha \neq \frac{\pi}{2}$

- The value of an inverse trigonometric function which lies in the range of principal branch is called the principal value of that inverse trigonometric function.
- The function $\sin^{-1}x$ is defined if $-1 \leq x \leq 1$; if α be the principal value of $\sin^{-1}$x then $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.
- The function $\cos^{-1}x$ is defined if $-1 \leq x \leq 1$; if α be the principal value of $\cos^{-1}x$ then $0 \leq \alpha \leq \pi$.
- The function $\tan^{-1}x$ is defined for any real value of x i.e. $-\infty < x < \infty$; if α be the principal value of $\tan^{-1}x$ then $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.
- The function $\cos^{-1}x$ is defined when $-\infty < x < \infty$ if α be the principal value of $\cos^{-1}x$ then $0 < \infty < \pi$.

- The function $\sec^{-1}x$ is defined when $|x|\geq 1$; if α be the principal value of $\sec^{-1}x$ then $0\leq\alpha\leq\pi$ and $\alpha \neq \frac{\alpha}{2}$.

- The function $\text{cosec}^{-1}x$ is defined if $|x|\geq 1$; if α be the principal value of $\text{cosec}^{-1}x$ then $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ and $\alpha\neq 0$.

- Properties of inverse trigonometric functions :

i) $\sin^{-1}(\sin x) = x$; $\sin(\sin^{-1}x) = x$

$\cos^{-1}(\cos x) = x$; $\cos(\cos^{-1}x) = x$

$\tan^{-1}(\tan x) = x$; $\tan(\tan^{-1}x) = x$

$\text{cosec}^{-1}(\text{cosec}\,x) = x$; $\text{cosec}(\text{cosec}^{-1}x) = x$

$\sec^{-1}(\sec x) = x$; $\sec(\sec^{-1}x) = x$

$\cot^{-1}(\cot x) = x$; $\cot(\cot^{-1}x) = x$

ii) $\sin^{-1}\left(\frac{1}{x}\right) = \cos ec^{-1}x;$ $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x, x > 0;$ $\cos ec^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x$

$\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x;$ $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}x$

iii) $\sin^{-1}(-x) = -\sin^{-1}x$; $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$; $\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$; $\cot^{-1}(-x) = \pi - \cot^{-1}x$

iv) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}, \; if \;\; xy < 1$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}, \; if \;\; xy > -1$

$\tan^{-1}x - \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), \; xy > 1, \; x, y > 0$

vi) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=2\tan^{-1}x$

vii) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$

$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$

Exercise-2

Section-A

a] OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

Multiple choice type questions : Choose the correct option

1) The principal value of $\cot^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ is

a) $\dfrac{\pi}{6}$ b) $\dfrac{\pi}{4}$ c) $\dfrac{\pi}{3}$ d) $\dfrac{\pi}{2}$

2) The value of $\cos\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ is

a) $n\pi-(-1)^n\dfrac{\pi}{3}$ b) $n\pi+(-1)^n\dfrac{\pi}{3}$ c) $n\pi+(-1)^n\dfrac{\pi}{6}$ d) $n\pi$

3) The general value of $\sin^{-1}\left(\dfrac{3}{5}\right)$ is

a) $\dfrac{5}{4}$ b) $\dfrac{4}{5}$ c) $-\dfrac{5}{4}$ d) $-\dfrac{4}{5}$

4) The value of $\sin\left(\sin^{-1}\dfrac{1}{3}+\sec^{-1}3\right)+\cos\left(\tan^{-1}\dfrac{1}{2}+\tan^{-1}2\right)$ is

a) –1 b) 0 c) 1 d) 2

5) The general value of $\cot^{-1}\left(-\sqrt{3}\right)$ is

a) $n\pi+\dfrac{5\pi}{6}$ b) $n\pi-\dfrac{5\pi}{6}$ c) $n\pi+\dfrac{\pi}{6}$ d) $n\pi-\dfrac{\pi}{6}$

6) If $2\tan^{-1}x = \sin^{-1}k$ then the value of k is

a) $\frac{1-x^2}{1+x^2}$ b) $\frac{2x}{1-x^2}$ c) $\frac{2x}{1+x^2}$ d) $\frac{2x^2}{1+x^2}$

7) State which of the statement is false ?

a) The formula $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$ holds when $|x| \geq 1$

b) $\sin^{-1}\cos\tan^{-1}\sqrt{3}$ represents an angle.

c) If $\cos^{-1}\frac{1}{\sqrt{5}} = \theta$, the the value of $\operatorname{cosec}^{-1}\sqrt{5}$ will be $\left(\frac{\pi}{2} - \theta\right)$.

d) If $\sin^{-1}x = \theta$, then the value of $\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ is also θ.

8) State which of the following is the value of $\left(\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2}\right)$?

a) $\frac{5\pi}{6}$ b) $\frac{\pi}{3}$ c) $\frac{2\pi}{3}$ d) $\frac{\pi}{2}$

9) If $\sec^{-1}x = \text{cosec}^{-1}y$, state which of the following is the value of $\left(\cos^{-1}\frac{1}{x} + \cos^{-1}\frac{1}{y}\right)$?

a) π b) $\frac{2\pi}{3}$ c) $\frac{5\pi}{6}$ d) $\frac{\pi}{2}$

10) If $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$, state which of the following is the value of x ?

a) 1 b) $\frac{1}{2}$ c) $\frac{1}{\sqrt{2}}$ d) $\frac{\sqrt{3}}{2}$

b] Very short answer type questions :

1) Find the principal value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

2) Find the range of general values of $\sec^{-1}(-2)$.

3) Find the value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sqrt{3}\right)$

4) Find the value of $\sec^2\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^2\operatorname{cosec}^{-1}\left(\sqrt{2}\right)$

5) Find the value of $\cot\left(\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}}\right)$

6) Show that $\cos^{-1}\frac{4}{5}+\cot^{-1}\frac{5}{3}=\tan^{-1}\frac{27}{11}$

7) If $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$ then find the value of $\cot^{-1}x+\cot^{-1}y$.

8) If $\text{Cos}\,(\tan^{-1}x+\cot^{-1}\sqrt{3})=0,$ then find the value of x.

9) Find the value of $\tan\frac{1}{3}\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)$ $x>0$?

10) Find the value of $\tan\cot^{-1}\left(-\frac{4}{3}\right)$

11) If $3\tan^{-1}x+\cot^{-1}x=\pi$ then find the value of x.

12) Find the domain of the functions $\cos^{-1}(2x-1)$.

Section-B

Short answer type questions : [3 marks for each question]

1) Find the value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$.

2) Prove that $\text{Sin}^{-1}\cos(\sin^{-1}x)+\text{cox}^{-1}\sin(\text{cox}^{-1}x)=\frac{\pi}{2}$.

3) Show that $4(2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7})=\pi$.

4) Show that $\frac{1}{2}\tan^{-1}x=\cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

5) Show that, $2\tan^{-1}\frac{1+x}{1-x}-\cos^{-1}\frac{1-x^2}{1+x^2}=\frac{\pi}{2}$

6) Prove that $\cos^{-1}\frac{1}{\sqrt{5}}+\cos^{-1}\frac{2}{\sqrt{5}}=\frac{\pi}{2}$

7) Prove that $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$

8) Prove that $\tan^{-1}a+\cot^{-1}b=\cot^{-1}\frac{b-a}{1+ab}$

9) Solve : $2\sin^{-1}x=\cos^{-1}x$.

10) Solve : $2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$

Section-C

Long answer type questions : [4/6 marks for each question]

1) Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

2) Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

3) Prove that $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$

4) Show that $2\tan^{-1}(-3)=-\frac{\pi}{2}+\tan^{-1}\left(\frac{-4}{3}\right)$

5) Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$

6) Prove that $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$

7) Show that $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$

8) Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\sin^{-1}\frac{1}{\sqrt{5}}$

9) Solve :

$\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$ and $\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$

10) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$ and $x+y+z=\sqrt{3}$, then show that $x=y=z$.

11) If $\sec\theta-\operatorname{cosec}\theta=\frac{4}{3}$, show that $\theta=\frac{1}{2}\sin^{-1}\frac{3}{4}$

12) Prove that $\frac{1}{2}\cos^{-1}\left(\frac{5\cos x+3}{5+3\cos x}\right)=\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)$

13) Prove that $\tan^{-1}\frac{1-x}{1+x}-\tan^{-1}\frac{1-y}{1+y}=\sin^{-1}\frac{y-x}{\sqrt{(1+x^2)(1+y^2)}}$

14) If $2\cos4\theta+9\cos2\theta-7=0$, then show that, $\theta=\frac{1}{2}\cos^{-1}\frac{3}{4}$.

15) Solve : $\sin^{-1}\frac{ax}{c}+\sin^{-1}\frac{bx}{c}=\sin^{-1}x$, where $a^2+b^2=c^2$ and $c\neq 0$.

ANSWERS

Section-A

a) Multiple choice questions :

1) c 2) b 3) b 4) c 5) d 6) c
7) d 8) c 9) d 10) d

b) Very short answer type questions :

1) $\frac{5\pi}{6}$ 2) $2n\pi \pm \frac{2\pi}{3}$, where $n=0, \pm 1, \pm 2$......... 3) π 4) 5

5) 0 7) $\frac{\pi}{5}$ 8) $x = \sqrt{3}$ 9) $\frac{1}{\sqrt{3}}$ 10) $-\frac{3}{4}$ 11) $x=1$

12) $0 \le x \le 1 \left(i.e.\ x \in [0,1]\right)$.

Section-B

Short answer type questions :

1) $-\frac{\pi}{10}$ 9) $\frac{1}{2}$ 10) $2-\sqrt{3}$

Section-C

Long answer type questions :

1) $-\frac{\pi}{12}$ 2) 0 9) $x=\frac{1}{2}$ 15) $\frac{\pi}{4}$

CHAPTER- 3

MATRICES

Important points and Results :

- A matrix is a rectangular array of numbers arranged in rows and columns. If m.n numbers are arranged in rows and columns. If m.n numbers are arranged in a rectangular array of m rows and n columns; it is called a **matrix of order m by n** (written as m×n) or a matrix of **dimension m×n** or a matrix of **types m×n**. A matrix of order m×n is usually written as follows :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots\ldots a_{2n} \\ a_{31} & a_{32} & a_{33}\ldots\ldots a_{3n} \\ & & \\ a_{m1} & a_{m2} & a_{m3}\ldots\ldots a_{mn} \end{bmatrix}$$

The numbers a_{11}, a_{12} etc. constitutting a matrix are called the elements or entries.

- Three different symbols are commonly used to enclose the elements forming a matrix :

$(\ )$; $[\ ]$; $\|\ \|$

Rectangular matrix :

If m.n numbers are arranged in a rectangular array of m rows and n columsn and if $m \neq n$, then array is called a **rectangular matrix** of order $m \times n$.

Example : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

- **Square matrix :**

An array of m^2 numbers arranged in the form of a square in m rows and m columns is called a square matrix of order $m \times n$.

Example : $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

- **Row matrix and column matrix :**

A matrix having only one row is called a **row matrix** or **row vector.** *Example :* $[1 \quad 2 \quad 3]$

A matrix containing only one column is called a column matrix or column vector.

Example : $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- **Null or zero matrix :**

  If all the elements of a matrix are zero, the matrix is called a null matrix or zero, the matrix is called a null matrix or zero matrix and is denoted by 0.

  Example :

  $$[0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Diagonal matrix, scalar matrix :**

  If the elements in the leading diagonal of a square matrix are non-zero and all the rest elements are zero, the matrix is called a diagonal matrix.

  And if the elements in the leading diagonal of a diagonal matrix are same quantities, the matrix is called a scalar matrix.

- **Identity or unit matrix :**

  If the elements in the leading diagonal of a square matrix are all unity (i.e. one) and all the rest elements are zero, the matrix is called an identity matrix or unit matrix and is usually denoted by the letter I.

- **Equality of matrices :**

  Two matrices are said to be equal if and only if they are of the same order and their elements in the corresponding positions are equal. The equality of two matrices A and B is denoted by A = B.

- **Transpose of a matrix :**

  Let A be a matrix of order m×n; then the matrix of order m×n obtained by interchanging the rows and columns of A is called the transpose of the matrix A and is donoted by A' or A^T.

- **Symmetric and Skew-symmetric matrices :**

  A square matrix A is said to be symetric if $A^T = A$; the matrix A is said to be skew-symmetric of $A^T = -A$ where A^T is the transpose of A.

- **Scalar multiplication of matrix :**

  The product of a matrix A and a scalar K is the matrix whose elements are obtained y multiplying each elements of A by the scalar K and is denoted by KA.

- **Addition and Substraction of two matrices :**

  Addition and Substraction of two matrices A and B are defined if and only if they are of the same order. The difference of A and B is defined as the addtion of negatives of B and A i.e. A–B = A+(–B) = A+(–1)B.

- If A, B and C are three matrices of the same order then

  i) A+B = B+A

  ii) (A+B)+C = A+(B+C)

iii) K(A+B) = KA+KB, where K is a scalar

iv) A+0 = 0+A = A

v) A+(–A) = (–A)+A = 0 where 0 is the null matrix.

vi) If A+C = B+C, then A=B.

- **Multiplication of matrices :**

  The product AB of two matrices A and B is defined if the numbers of columns in A is equal to the number of rows in B. If A=$[a_{ij}]_{m\times p}$ and B=$[b_{ij}]_{p\times n}$ martices, then the product matrix AB is order $m\times n$.

- If A, B and C are three matrices, then

  i) In general, AB≠BA i.e. multiplicatin of matrices, in general; is not commutative.

  ii) (AB)C = A(BC) provided the products involved are defined.

  iii) A(B+C) = AB+AC provided the sums and products involved are defined.

  iv) CA = CB does not necessarily imply A = B.

  v) A.O = O.A = O, where O is the null matrix.

  vi) A.I = I.A. = A, where I is the unit matrix.

  vii) A ≠ O, B ≠ O can imply AB = O, where O is the null matrix.

- A square matrix of order n is said to be orthogonal if $A.A^T = A^T.A = I$.

- If A^T and B^T are transpose of the matrices A and B respectively, then

  i) $\left(A^T\right)^T = A$

  ii) $(A+B)^T = A^T+B^T$

  iii) $(A–B)^T = A^T–B^T$

  iv) $(AB)^T = B^T.A^T$

- A square matrix A can be expressed as the sum of symmetric and a skew-symmetric matrices. That is $A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$ where $\frac{1}{2}\left(A + A^T\right)$ is symmetric matrix and $\frac{1}{2}\left(A - A^T\right)$ is skew symmetric matrix.

- **Singular and non-singular matrices :**

  The determinant formed by the elements of a same matrix A is called the determinant of the matrix A and is denoted by |A| or detA. If det A = 0; the matrix A is called singular and if det A ≠ 0, the matrix A is called non-singular.

- **Adjoint or adjubate matrix :**

  The adjoint matrix of a given square matrix A is the transpose of the matrix whose elements are the cofactors of the elements in |A| and and is denoted by adjA. To find the adjoint matrix of a square matrix A, the co-factors of the elements of successive rows of |A| are written in the corresponding columns.

- If A be a non-singular matrix (i.e. $|A| \neq 0$), then the matrix $\frac{adjA}{|A|}$ is called the reciprocal matrix of A.

- **Inverse of a matrix :**

  If A and B are two square matrices of the same order such that AB = BA = I; where I is the unit matrix of the same order as A or B, then either B is called the inverse of the matrix A and is denoted by $B = A^{-1}$ or A is called the inverse of the matrix B and is denoted by $A = B^{-1}$.

- The inverse of a square matrix A exist of $|A| \neq 0$ i.e. if A is non-singular then, $A^{-1} = \frac{adj A}{|A|}$.

- **Properties of Inverse matrix :**

  i) The inverse of a given square matrix, if it exists, is unique.

  ii) The inverse of the inverse of a matrix is the matrix itself. i.e. for the square matrix A, we have $\left(A^{-1}\right)^{-1} = A$, when A^{-1} exists (i.e. when A is invertible).

  iii) The transpose of the inverse is equal to the inverse of the transpose of the given matrix i.e. for the matrix A, $\left(A^{-1}\right)^{T} = \left(A^{T}\right)^{-1}$.

  iv) If A and B are two non-singular square matrices and A^{-1} and B^{-1} are then respective inverses, then $(AB)^{-1} = B^{-1}A^{-1}$.

- If B is the inverse matrix of A, then A and B both are inverse matrix.

- The inverse of a matrix A and the reciprocal matrix of A are equal to each other.

- The unit matrix is the inverse of itself, i.e. $I^{-1} = I$.

- The zero matrix has no inverse.

- **Elementary Operation (Transformation) of a matrix :**

  The following tree types of transformation applied on the rows (or columns) of a matrix are known as elementary transormations.

  i) Interchange of any two rows (or columns). The interchange of i-th row (or column) with j-th row (or column) of a matrix is denoted by $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$).

  ii) Multiplication of the elements of any row (or column) of a matrix by a non-zero scalar. The multiplication of each element of the i-th row (or column) of a matrix by a non-zero scalar K is denoted by $R_i \to KR_i$ (or $C_j \to KC_i$)

  iii) To multiply each element of a row (or column) of a given matrix by a non-zero scalar and to add them to the corresponding elements of any other row (or column). Thus $R_i \to R_i + KR_j$ (Or $C_i \to C_j + KC_j$)

Exercise-3

Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1] Multiple choice type questions :

i) If AB=A and BA=B, then B^2 is equal to

a) B b) A c) –B d) B^3

ii) If A is 3×4 matrix and B is a matrix such that $A'B$ and $B'A$ are both defined; then B is of the type

a) 4×4 b) 3×4 c) 4×3 d) 3×3

iii) If $A = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$ is a symmetric matrix, then the value of x is

a) 4 b) 3 c) –4 d) –3

iv) If $A=[a_{ij}]_{2\times2}$, where $a_{ij}=i+j$; then A is equal to

a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

v) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ than adj (adjA) =

a) A b) I c) A^2 b) A'

vi) Suppose A is a 4×4 matrix such than |A|=4 and |adjA|=8K, then value of K will be

a) 5 b) 8 c) 3 d) 2

vii) If A be a proper orthogonal matrix, then –

a) |A|=0 b) |A|=1 c) |A|=2 d) |A|=3

viii) If A is an invertible matrix of order 3 and |A| = 5, then the value of |adjA| is equal to –

a) 20 b) 21 c) 24 d) 25

ix) If A is a non-singular matrix of order 3 and x is a real number such than det(xA)=|x|det(A) then the value of x is –

a) 0 or 1 b) 0 or –1 c) 1 or –1 d) 0 or ±1

x) The necessary and sufficient conditon that any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of order 2×2 has an inverse is –

a) $ab–cd = 0$ b) $ad–bc \neq 0$ c) $ac–bd \neq 0$ d) $ad+bc \neq 0$

2] Very short answer type questions :

i) If $A=\begin{bmatrix}-2 & 2 & 1\\ 0 & 4 & 5\\ -2 & 6 & 6\end{bmatrix}$, does A^{-1} exist ?

ii) Show that, $A=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ is a proper orthogonal matrix. Hence find A^{-1}.

iii) If $A=\begin{pmatrix}3 & 1\\ 0 & 2\end{pmatrix}$ show that $\left(A^{T}\right)^{-1}=\left(A^{-1}\right)^{T}$. where A^T is the transpose of A.

iv) Show that the matrix $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$ satisfies the equation $x^2-6x+17=0$. Hence find A^{-1}.

v) If $A=\begin{pmatrix}22 & 13\\ 17 & 8\end{pmatrix}$; then find $B=A+A^T$ and show that $B^T=B$.

vi) If A be a 2×2 matrix such that $A^2=A$, then show that $(I-A)^2=I-A$, where I is the unit matrix of order 2×2.

vii) If for a matrix A, $A=A^{-1}$; then show that $A(A^3+I)=A+I$. (I is the unit matrix)

viii) If $P=\begin{bmatrix}1 & \alpha & 3\\ 1 & 3 & 3\\ 2 & 4 & 4\end{bmatrix}$ is the adjoint of 3×3 matrix A and $|A|=4$, then find the value of α.

ix) If $A=\begin{bmatrix}5a & -b\\ 3 & 2\end{bmatrix}$ and $A\,\mathrm{adj}A=AA^T$, then find the value of $5a+b$.

x) Write 2×2 matrix which is both symmetric and skew-symmetric.

Section-B

3] Short answer type questions : [3 marks for each question]

i) If $A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$, then find α satisfying $0<\alpha<\frac{\pi}{2}$ when $A+A^T=\sqrt{2}I_2$; where A^T is transpose of A.

ii) If A is a square matrix such that $A^2=I$, then find the simplified value of $(A-I)^3+(A+I)^3-7A$.

iii) If $A=\begin{bmatrix}P & 2\\ 2 & P\end{bmatrix}$ and $\left|A^3\right|=125$; then find the value of P.

iv) If A is a 3×3 non-singular matrix such that $AA^T=A^TA$ and $B=A^{-1}A^T$; then find BB^T.

v) If $A=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$ and $A^2-4A+3I=0$, where I is the unit matrix of order 2 then find A^{-1}.

vi) Find the inverse of the given matrix by unsing elementary column operations : $\begin{bmatrix}-3 & 5\\ -4 & 5\end{bmatrix}$

vii) If $A=\begin{pmatrix}4 & 5\\ 2 & 1\end{pmatrix}$ show that, $6A^{-1}+5I=A$

viii) If $A=\begin{pmatrix}3 & 1\\ 7 & 5\end{pmatrix}$ and $A^2=-xI+yA$, find x and y where I is unit matrix of order 2.

ix) If AB=BA for any two square matrices, then prove by mathematical induction that $(AB)^n=A^nB^n$.

x) If $A=\begin{bmatrix}2 & q\\ o & i\end{bmatrix}$ and $A^8=\begin{pmatrix}x & yq\\ o & i\end{pmatrix}$, then find x – y.

Section-C

Long answer type questions : [4/6 marks for each question]

i) If $A^{-1}=\begin{bmatrix}1 & 3 & 2\\ -3 & -3 & -1\\ 2 & 1 & 0\end{bmatrix}$, find A^{-1}

ii) If $A=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$, find AA^T; hence find A^{-1}.

iii) Express the following matrix as a sum of a symmetric and a skew symmetric matrices and verify your result : $\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}$

iv) If $A=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$, find *adj*A and verify that $A(adjA)=(adjA)A=|A|=I_3$.

v) Let $A=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}$. If U_1 and U_2 are column matrices such that $AU_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $AU_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$; then find U_1+U_2.

vi) If $P=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \left(\frac{-1}{2}\right) & \frac{\sqrt{3}}{2}\end{bmatrix}$, $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $Q=PAP^T$, then find $P^TQ^{2015}P$.

vii) Show that the matrix $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$ satisfies the equation $A^2-4A-5I_3=0$ and hence find A^{-1}.

viii) a) If the matrix $A=\begin{bmatrix}3 & 1 & x\\ y & 7 & -2\\ 5 & z & -4\end{bmatrix}$ is a symmetric matrix, find the values of x, y and z.

b) If the matrix $A=\begin{bmatrix}0 & 2 & x\\ y & 0 & z\\ 6 & -5 & 0\end{bmatrix}$ is a skew-symmetric matrix, find the values of x, y and z.

ix) If $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, show that $(aA+bB)(aA-bB)=(a^2+b^2)A$.

x) The matrix $R(t)$ is given by $R(t)=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$. Show that $R(s).R(t)=R(s+t)$.

xi) Given that $A=\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}$ and $A(adjA)=K\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, find the value of K.

xii) Using elementary transformations, find the inverse of the matrix : $\begin{bmatrix}1 & 3 & -2\\ -3 & 0 & -1\\ 2 & 1 & 0\end{bmatrix}$

ANSWERS

Section-A

1) i) a ii) b iii) c iv) d v) a vi) b
vii) b viii) d ix) a x) b

2) i) yes ii) $A^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ iv) $\frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$

v) $B = A + A^T = \begin{pmatrix} 44 & 30 \\ 30 & 16 \end{pmatrix}$ vii) $\alpha = 11$ ix) 5 x) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Section-B

3) i) $\alpha = \frac{\pi}{4}$, which is satisfying $0 < \alpha < \frac{\pi}{2}$ ii) A iii) ± 3 iv) $BB^T=I$

vi) $\frac{1}{5}\begin{pmatrix} 5 & -5 \\ 4 & -3 \end{pmatrix}$ viii) x=8, y=8

Section-C

4) ii) $A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$ v) $U_1+U_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ vi) $\begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$

vii) $\frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

viii) a) $x = 5,\ y = 1, z = 2$
b) $x = 6,\ y = -2, z = 5$

xii) $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$

CHAPTER- 4

DETERMINANTS

Important points and Results :

- If A=[a_{ij}] is a square matrix of order $n \times n$, then the determinant associated with matrix A is an expression which is denoted by |A| or det A or, $|a_{ij}|$ (i=1,2,3.........n; j=1,2,3,.....n).

  $$\text{i.e. } |A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots & a_{nn} \end{vmatrix}$$

  Hence $|A| = |a_{ij}|$ is a determinant with n rows an n columns and is known as a determinant of order n.

- **Expansion of a 2nd order determinant :**

  $D_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ (Product of the elements along the leading diagonal) – (the product of the elements along the secondary diagonal) $= a_1 b_2 - a_2 b_1$

- **Expansion of a 3rd order determinant :**

  $$D_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_3 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

  [expanding by the 1st row].

- Determinant of a matrix $A=[a_{ij}]_{i \times j}$ is given by $|a_{ij}|=a_{ij}$

- **Minors :**
  Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.
  Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n-1$.

- **Cofactors :**
  Cofactor of an element a_{ij}, denoted by A_{ij} is defined by $A_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is minor of a_{ij}.

- If A_1, B_1, C_1 etc. are the respective cofactos of the elements a_1, b_1, c_1 etc. in D_2 then

$$a_iA_j + b_iB_j + c_iC_j = \begin{cases} 0 & when\ i \neq j \\ D_2 & when\ i = j \end{cases}$$

and $a_1A_1 + a_2A_2 + a_3A_3 = b_1B_1 + b_2B_2 + b_3B_3 = c_1C_1 + c_2C_2 + c_3C_3 = D_2$
and $a_1B_1 + a_2B_2 + a_3B_3 = a_1C_1 + a_2C_2 + a_3C_3 = 0$ etc.

- **Properties of a determinant :**
For any square matrix A, the |A| satisfy following properties :

1. $|A'| = |A|$, where A' = transpose of A.
2. If we interchange any two rows (or columns) then sign of determinant changes.
3. If any two rows or any two columns are identical or proportional, then value of determinant is zero.
4. If we multiply each element of a row or a column of a determinant by constant K, then value of determinant is multiplied by K.
5. Multiplying a determinant by K means, multiply elements of only one row (or one column) by K.
6. If $[a_{ij}]_{3\times3}$, then $|K.A| = K^3|A|$
7. If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.
8. If to each element of a row or a column of a determinant the equimultiples of corresponding elements of order rows or columns are added, then value of determinant remains same.

- Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\text{area } (\Delta ABC) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Condition for collinearity of three points :
area $(\Delta ABC) = 0$

- Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and denoted by M_{ij}.

- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $adj A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

  Where A_{ij} is cofactor of a_{ij}.

- $A(adj A) = (adj A)A = |A|I$, where A is square matrix of order n.

- A square matrix A is said to be singular or non-singular according as $|A|=0$ or $|A|\neq 0$.
- If $AB=BA=I$ where B is square matrix, then B is called inverse of A.

  Also $A^{-1}=B$ or $B^{-1}=A$ and hence $\left(A^{-1}\right)^{-1}=A$.
- A square matrix A has inverse if and only if A is non-singular.
- $A^{-1}=\dfrac{1}{|A|}(adjA)$
- If $a_1x+b_1y+c_1z=d_1$

  $a_2x+b_2y+c_2z=d_2$

  $a_3x+b_3y+c_3z=d_3$

  then these equations can be written as $AX=B$, where

  $$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$
- Unique solution of equation $AX=B$ is given by $X=A^{-1}B$, where $|A|\neq 0$.
- A system of equation is consistent or in cosnsitant according as its solution exists or not.
- For a square matrix A in matrix equation $AX=B$
  - i) $|A|\neq 0$, there exists unique solution.
  - ii) $|A|=0$ and $(adjA)B\neq 0$, then there exists no solution.
  - iii) $|A|=0$ and $(adjA)B=0$, then there system may or may not be consistent.
- The adjoint of a given determinant D is the determinant whose elements are co-factors of the corresponding elements of D and is denoted by D'.
- $D' = D^2$

Exercise-4

Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

Multiple choice type questions : (Choose the correct option)

1) The value of $\begin{vmatrix} 1 & -3 & 5 \\ -2 & 4 & 3 \\ 4 & 5 & 6 \end{vmatrix}$ is

a) 193 b) –193 c) 190 d) –192

2) If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_1, B_1, C_1 etc are the respective co-factors of the elements a_1, b_1, c_1 etc.

then D will be –

a) $a_2C_2+b_2C_2+c_2C_2$ b) $c_1C_1+c_2C_2+c_3C_3$ c) $a_2A_1+b_2B_1+c_2C_1$ d) $a_1B_1+a_2B_2+a_3B_3$

3) The value of the determinant formed by the elements of an identity matrix is always

a) 1 b) –1 c) 2 d) –2

4) If two rows (or two columns) of a determinant are identical then the value of the determinent is –

a) 1 b) 2 c) –1 d) None of these

5) Given that $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ then the other two roots are –

a) –2, 7 b) 2, –7 c) 2, 7 d) –2, –7

6) The value of $\begin{vmatrix} 0 & 2 & -3 \\ 1 & 2 & 4 \\ -2 & 3 & 2 \end{vmatrix} \times \begin{vmatrix} 2 & 0 & -3 \\ 3 & 1 & 0 \\ 0 & 4 & -2 \end{vmatrix}$ is

a) –1640 b) 1480 c) 1640 d) 1380

7) The value of $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$, where ω is an imaginary cube root of unity.

a) 1 b) 3 c) –3 d) –1

8) If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$ then the value of x is

a) 3 b) ± 3 c) ± 6 d) 6

9) If the area of a triangle with vertices (–3,0), (3,0) and (0,K) is 9 sq units. Then the value of K will be

a) 9 b) 3 c) –9 d) 6

10) If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to

a) 0 b) –1 c) 1 d) none of these

11) If cos2θ=0, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$ is equal to

a) $-\frac{1}{2}$ b) $\frac{1}{2}$ c) 2 d) –2

12) The maximum value of $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$ is (where, θ is real number)

a) $\frac{1}{2}$ b) $\frac{\sqrt{3}}{2}$ c) $\sqrt{2}$ d) $\frac{-2\sqrt{3}}{4}$

Very short answer type questions :

1) Find the value of $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

2) If $\begin{vmatrix} 32 & 24 & 16 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix} = K \begin{vmatrix} 1 & 3 & 2 \\ 2 & 3 & 5 \\ 1 & 5 & 3 \end{vmatrix}$ then find the value of K.

3) Find the value of $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$

4) Without expanding find the value of $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$.

5) If $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 4 & 0 & 1 \end{bmatrix}$, then find the value of λ, if $|5A| = \lambda |A|$.

6) Evalute : $\begin{vmatrix} \cos 15^0 & \sin 15^0 \\ \sin 75^0 & \cos 75^0 \end{vmatrix}$

7) If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, the write the value of x.

8) If A_{ij} is the co-factor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32}.A_{32}$.

9) If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, then write the minor of the element a_{23}.

10) Find the minor of the element of second row and third column (a_{23}) in the following determinant.

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

11) Find the cofactor of a_{12} in the following :

$$\begin{vmatrix} 1 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

12) For what value of x, the following matrix is singular ?

$$\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$$

13) Evalute : $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

14) Evalute : $\Delta = \begin{vmatrix} 4 & a & b+c \\ 4 & b & c+a \\ 4 & c & a+b \end{vmatrix}$

15) If $A = \begin{vmatrix} 3 & 10 \\ 2 & 7 \end{vmatrix}$, then write A^{-1}.

Section-B

Short answer type questions : (2 marks each)

1) Find the area of the triangle with vertices at the points (2,7), (1,1), (10,8).

2) Using properties of determinants, prove that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

3) Show that the points A(3,8), B(–4,2) and C(10,14) are collinear.

4) A(–2,0), B(0,4) and C(0,K) be three points such that area (ΔABC)=4 sq units. Find the value of K.

5) Find the equation of the line joining the points A(2,4) and B(6,12), using determinant.

6) If the points (a,b), (a',b') and $(a-a',b-b')$ are collinear, show that $ab' = a'b$.

7) Find the area of the triangle whose vertices are $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$.

Section-C

Long answer type questions : (4 or 6 marks for each questions)

1) By using properties of determinant, prove that $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

2) Evaluate $\begin{vmatrix} Cos^2\theta & Sin\theta\, Cos\theta & -Sin\theta \\ Sin\theta\, Cos\theta & Sin^2\theta & Cos\theta \\ Sin\theta & -Cos\theta & 0 \end{vmatrix}$

3) Show that $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$ where *a, b, c* are in A.P.

4) Solve the following system of equation s, using matvices $\mathrm{x}+3\mathrm{y}=3,\ \mathrm{y}+3\mathrm{z}=7,\ \mathrm{z}+4\mathrm{x}=2$.

5) Using properties of determinant, prove that $\begin{vmatrix} 3a & -a+b & -a+c \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

6) Without expanding prove that $\begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix} = 0$

7) Using properties of determinants prove the following :

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$$

8) Using properties of determinant, solve for *x*

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

9) Using properly of determinant, prove the following

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$$

10) Using properties of determinant, prove the following

$$\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = \left(1-x^2\right)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

11) Using properties, prove that $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$

12) Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c_3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

13) Show that $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

14) Prove the following using properties of determinant

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\left(3abc - a^3 - b^3 - c^3\right)$$

15) If *a, b, c* are positive and unequal, then show that the following determinant is negative

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

16) If a, b and c are all positive and distinct, show that $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ has a negative value.

17) Prove the following using properties of determinant

$$\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix} = \left(x^4-1\right)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$$

18) Using properties of determinant, prove the following

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$$

19) Using properties of determinant, prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = ab+bc+ca+abc$.

20) Prove using properties of determinant $\begin{vmatrix} y+5 & y & y \\ y & y+5 & y \\ y & y & y+5 \end{vmatrix} = 25(3y+5)$

21) Solve the following system of linear equations by matrix inversion method.

$4x + 2y + 3z = 2$; $x + y + z = 1$; $3x + y - 2z = 5$

22) If a, b, c are realnumber, then prove that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$,

where, ω is a complex cube root of unity.

23) In a triangle ABC if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$ then prove that ΔABC is an isoscales triangle.

24) Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then find $\lim_{t\to 0}\frac{f(t)}{t^2}$.

ANSWERS

Section-A

Multiple choice type questions :

1((b)	2(b)	3(a)	4(d)	5(c)	6(c)	7(b)
8(c)	9(b)	10(a)	11(b)	12(a)		

Very short answer type questions :

1) 0	2) 32	3) −8	4) 0	5) λ= 125	6) 0
7) 2	8) 110	9) 7	10) 13	11) 46	12) x=3
13) $a^2+b^2+c^2+d^2$		14) 0	15) $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$		

Section - B

Short answer type questions :

1) $\frac{47}{2}$ sq. unit 4) K=0, or K=8 5) $y = 2x$

7) $a^2|(t_1 - t_2)(3_2 - t_3)(t_3 - t_1)|$ sq. unit.

Section-C

Long answer type questions :

2) 1 4) x = 0, y = 1, z = 2 8) x = 0 or x = 3a 21) $x = \frac{1}{2}$, $x = \frac{3}{2}$, $z = -1$

24) 0

CHAPTER- 5

CONTINUITY AND DIFFERENTIABILITY

Important points and Results :

- **Continutity of a function of a point :**

  Let f be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

  More elaborately, if the left hand limit, right hand limit and the value of the function of at $x = c$ exist and are equal to each other, i.e.

  $$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

  then f is said to be continuous at $x = c$.

- **Geometrical meaning of continuity :**

  i) Function f will be coninuous at $x = c$ if there is no break in the graph of the function at the point $(c, f(c))$.

  ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.

- **Continuity in an interval :**

  i) f is said to be continuous in an open interval (a,b) if it is continuous at every point in this interval.

  ii) f is said to be continuous in the closed interval $[a,b]$ if

    1) f is continuous in (a, b).

    2) $\lim_{x \to a^+} f(x) = f(a)$

    3) $\lim_{x \to b^-} f(x) = f(b)$

- **Discontinuity :**

  The function f will be discontinuous at $x = c$ in any of the following cases :

  i) $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist but are not equal.

  ii) $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and are equal but not equal to $f(c)$.

  iii) $\lim_{x \to c} f(x)$ does not exist.

  iv) $f(c)$ is not defined.

- **Continuity of some of the common functions :**

Function $f(x)$	Interval in which f is continuous.
i) A constant function, i.e. $f(x) = c$	R
ii) The identity function, i.e. $f(x) = x$.	R
iii) The polynomial function, i.e, $f(x) = a_0x^n + a_1x^{n-1} + \dots\dots + a_{n-1}x + a_n$	R
iv) Modulus function, i.e. $f(x) = \lvert x \rvert$.	R
v) Exponential function, i.e. $f(x) = e^x$ or a^x.	R
vi) $f(x) = \sin x$ or $\cos x$	R
vii) Rational function i.e. $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$	R
viii) A logarithamic function, i.e. $f(x) = \log x$	$(0,\infty)$
ix) $f(x) = \tan x$, $\sec x$	$R - \left\{(2n+1)\dfrac{\pi}{2} : n \in z\right\}$
x) $f(x) = \cot x$, $\operatorname{cosec} x$	$R - \{n\pi : n \in z\}$
xi) $f(x) = \sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, $\cot^{-1}x$	In their respective domains.

- **Algebra of continuous functions :**

  Let $f(x)$ and $g(x)$ be two continuous function on their common domain and c be a real number in their domain. Then at $x = c$.

  i) $f + g$ is continuous.

  ii) $f - g$ is continuous.

  iii) $f.g$ is continuous.

  iv) $\dfrac{f}{g}$ is continuous.

  v) f^n, $n \in N$ is continuous.

- **Continuity of composite functions :**

  Let f and g be real valued functions such that (fog) is defined at c. If g is continuous at c and f is continuous at $g(c)$, then (fog) is continuous at c.

- **Differentiability :**

  Let f be a real function and c is a point in its domain. We say that f is differentiable at c if

  $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists and is finite.

  We denote this limit by $f'(c)$, called the derivative or differential coefficient of the function $f(x)$ at c.

  In other words, we say that a function f is differentiable at a point c in its domain if both $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$

or $\underset{h\to 0}{Lim}\dfrac{f(c-h)-f(c)}{-h}$, called left hand derivative, denoted by $Lf'(c)$, and $\underset{x\to c^{+}}{Lim}\dfrac{f(x)-f(c)}{x-c}$ or

$\underset{h\to 0}{Lim}\dfrac{f(c+h)-f(c)}{h}$, called right hand derivative, denoted by $Rf'(c)$, are finite and equal.

If $Lf'(c)\neq Rf'(c)$, we say that $f(x)$ is not differentiable at $x=c$.

- **Differentiability of a function on an interval :**

  i) A function $f(x)$ is said to be differentiable in an open interval (a,b) if it is differentiable at every point of (a,b).

  ii) A function $f(x)$ is said to be differentiable in a closed interval $[a,b]$ if it is differentiable in the open interval (a,b) and if $Rf'(a)$ and $Lf'(b)$ exist.

- Every differntiable function is continuous, but every continuous function need not be differentiable.

- **Geometrical meaning of derivative :**

  The derivative of $f(x)$ at $x=c$ is the slope of the tangent to the curve $y=f(x)$ at the point $(c, f(c))$.

- Following are some of the standard derivatives :

  i) $\dfrac{d}{dx}\left(\sin^{-1}x\right)=\dfrac{1}{\sqrt{1-x^2}},\ -1<x<1$ ii) $\dfrac{d}{dx}\left(\cos^{-1}x\right)=\dfrac{-1}{\sqrt{1-x^2}},\ -1<x<1$

  iii) $\dfrac{d}{dx}\left(\tan^{-1}x\right)=\dfrac{1}{1+x^2},\ -\infty<x<\infty$ iv) $\dfrac{d}{dx}\left(\cot^{-1}x\right)=\dfrac{-1}{1+x^2},\ -\infty<x<\infty$

  v) $\dfrac{d}{dx}\left(\sec^{-1}x\right)=\dfrac{1}{|x|\sqrt{x^2-1}},\ |x|>1$ vi) $\dfrac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)=\dfrac{-1}{|x|\sqrt{x^2-1}},\ |x|>1$

- **Derivatives of composite functions :**

  (Chain rule) : Let f be a real valued function which is a composite of two functions u and v; i.e. $f = vou$. Suppose $t = u(x)$ and if both $\dfrac{dt}{dx}$ and $\dfrac{dv}{dt}$ exist, we have $\dfrac{df}{dx}=\dfrac{dv}{dt}\cdot\dfrac{dt}{dx}$.

  This chain rule may be extended more than two functions.

- **Derivatives of implicit functions :**

  If the variables x and y are connected by a relation of the form $f(x,y) = 0$ and it is not possible to express y as a function x in the form $y=\phi(x)$, then y is said to be an implicit function of x. To find $\dfrac{dy}{dx}$ is such a case, we differentiate both sides of the given relation w.r.t x, keeping in mind that the derivative of $\phi(y)$ with respect to x is $\dfrac{d\phi}{dy}\cdot\dfrac{dy}{dx}$.

- **Logarithmic Differentiation :**

  The process of taking logarithms before differentiation is called logarithmic differentiation. When we differentiate certain special class of functions given in the form $y = f(x) = [u(x)]^{v(x)}$.

  Taking logarithm (to base e) of both the sides and then differentiate. The main point to be noted in this method is that $f(x)$ and $u(x)$ must always be positive as otherwise their logarithms are not defined.

- **Derivatives of functions in Parametric forms :**

  Sometimes x and y are separately given as functions of a single variable t (called a parameter) i.e., $x = f(t)$ and $y = g(t)$. In order to find derivative of function in such form by the following formula

  $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left(whenever \ \frac{dx}{dt} \neq 0 \right).$$

- **Derivative of a function with respect to another function :**

  Let $u = f(x)$ and $v = g(x)$ be two function of x, then to find derivative of $f(x)$ w.r.t $g(x)$, i.e., to find $\frac{du}{dv}$, we use the formula $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$.

- **Second order Derivative :**

  Let $y = f(x)$. Then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}$ is called the second order derivative of y w.r.t x. It is denoted by $y^{//}$ or y_2 a $f''(x)$.

  Rolle's Theorem :

  Let $f:[a,b] \rightarrow R$ be continuous on $[a,b]$ and differentiable on (a,b), such that $f(a) = f(b)$, where a and b are some real numbers. Then there exists at least one point c in (a,b) such that $f'(c) = 0$.

  Geometrically Rolle's theorem ensures that there is at least one point $(c, f(c))$ lying between $(a, f(a))$ and $(b, f(b))$ at which tangent is parallel to x-axis.

- **Mean Value Theorem :**

  Let $f:[a,b] \rightarrow R$ be a continuous function on $[a,b]$ and diffrentiable on (a,b). Then there exist at least one point c in (a,b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. Geometrically, Mean value theorem states that there exists at least one point c in (a,b) such that the tangent at the point $(c, f(c))$ is parallel to the secant joining the points $(a, f(a))$ and $(b, f(b))$.

Exercise - 5
Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) Multiple choice type questions :

i) The function $f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & x\neq 0\\ \dfrac{K}{2}, & x=0\end{cases}$ is continuous at $x=0$, then the value of K is

a) 3 b) 6 c) 9 d) 12

ii) The function $f(x)=[x]$, where $[x]$ denotes the greatest integer function, is continuous at

a) 3 b) –2 c) 1.5 d) 1

iii) The number of points at which the function $f(x)=\dfrac{1}{x-[x]}$ is not continuous is

a) 1 b) 2 c) 3 d) none of these

iv) The function $f(x)=\cot x$ is discontinuous on which set ?

a) $\{n\pi : n\in z\}$ b) $\{2n\pi : n\in z\}$ c) $\left\{(2n+1)\dfrac{\pi}{2} : n\in z\right\}$ d) $\left\{\dfrac{n\pi}{2} : n\in z\right\}$

v) The set of points where the function $f(x)=|x-2|\cos x$ is differentiable are

a) R b) R–{2} c) $(0, \infty)$ d) none of these

vi) If $f(x)=3x$ and $g(x)=\dfrac{x^3}{3}+3$, then which of the following can be a discontinuous function

a) $f(x)+g(x)$ b) $f(x)-g(x)$ c) $f(x).g(x)$ d) $\dfrac{f(x)}{g(x)}$

vii) The function $f(x)=\dfrac{5-x^2}{5x-x^3}$ is

a) discontinuous at only one point b) discontinuous at exactly two points.
c) discontinuous at exactly three points. d) none of these.

viii) If $f(x)=|x-2|$ and $g(x)=f\{f(x)\}$, then for $2<x<4$, the value of $g'(x)$ is

a) –1 b) 0 c) 1 d) none of these.

ix) If $f(x)=e^x\,\phi(x)$, $\phi(0)=1$, $\phi'(0)=3$, then the value of $f'(0)$ is

a) 0 b) 2 c) 4 d) none of these.

x) If $f(x)=log_x(logx)$, then the value of $f'(e)$ is

a) 0 b) e c) $\dfrac{2}{e}$ d) $\dfrac{1}{e}$

xi) If $y=f(x)$ is an odd differentiable function defined on $(-\infty,\infty)$ such that $f'(3)=-2$, then $f'(-3)$ equals

a) 4 b) 2 c) –2 d) 0

xii) If $x^2+y^2=1$ then

a) $yy'' - 2(y')^2 + 1 = 0$ b) $yy'' + (y')^2 + 1 = 0$

c) $yy'' + (y')^2 - 1 = 0$ d) $yy' + 2(y')^2 + 1 = 0$

xiii) If $f(x) = |\cos x - \sin x|$, then $f'\left(\pi/2\right)$ is

a) 1 b) -1 c) 0 d) none of these.

xiv) The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}\,]$ is

a) 1 b) -1 c) $\frac{3}{2}$ d) $\frac{1}{3}$

xv) The value of c in Mean value theorem for the function $f(x) = x + \frac{1}{x}$ in the interval [1, 3] is

a) 1 b) $\sqrt{3}$ c) 2 d) none of these.

2] Very short answer type questions :

i) If $f(x) = \begin{cases} \dfrac{\sin^{-1} x}{x}, & x \neq 0 \\ K, & x = 0 \end{cases}$ is continuous at $x = 0$, then find the value of K.

ii) Find the value of the derivative of $f(x) = |x-1| + |x-3|$ at $x = 2$.

iii) If $f(x) = \sqrt{x^2+9}$, then find the value of $\lim\limits_{x \to 4} \dfrac{f(x) - f(4)}{x-4}$.

iv) If $y = \sec^{-1}\left(\dfrac{x+1}{x-1}\right) + \sin^{-1}\left(\dfrac{x-1}{x+1}\right)$, $x>0$ then find $\dfrac{dy}{dx}$.

v) Differentiate $log(1+x^2)$ with respect to $\tan^{-1}x$.

vi) If $y = log_a x$, find $\dfrac{dy}{dx}$.

vii) If $f(x)$ is an even function, then write whether $f'(x)$ is odd or even.

viii) If $x = t^2$, $y = t^3$, then find $\dfrac{d^2y}{dx^2}$.

ix) If $y = \sin x^0$, then find $\dfrac{dy}{dx}$.

x) If $y = \sin^{-1}(\cos x)$, then find $\dfrac{d^2y}{dx^2}$.

Section-B

3] Short answer type questions : [3 marks for each question]

i) Prove that $f(x) = \sqrt{|x| - x}$ is continuous for all $x \geq 0$.

ii) Given $f(x) = \dfrac{1}{x-1}$. Find the points of discontinuity of the function $f\{f(x)\}$.

iii) Discuss the differntibility of $f(x) = x\,|\,x\,|$ at $x = 0$.

iv) If $x = e^{\frac{x}{y}}$, then prove that $\frac{dy}{dx} = \frac{x-y}{x\log x}$.

v) Differentiate each of the following functions w.r.t. x.

a) $3^{\sin^2 x}$ b) $\frac{5^x}{x^5}$ c) $\log\left(x+\sqrt{1+x^2}\right)$ d) $\log_{10}(\log\cos x)$

e) $\tan^{-1}\left\{\frac{\cos x}{1+\sin x}\right\}, 0 < x < \pi$ f) $\tan^{-1}\left(\frac{x}{1+6x^2}\right)$ g) $\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$

h) $\sin(x^x)$ i) $e^{x\log x}$

vi) If $f(x) = \sin x$, $g(x) = 2x$, $h(x) = \cos x$ and $\phi(x) = \{go(foh)\}.(x)$, then find $\phi''\left(\pi/4\right)$

vii) If $y = e^{nx}$, then find $\left(\frac{d^2y}{dx^2}\right).\left(\frac{d^2x}{dy^2}\right)$.

Section-C

Long answer type questions : (4 or 6 marks each)

4. Discuss the continuity of the following functions at the indicated points.

i) $$f(x) = \begin{cases} \frac{2|x|+x^2}{x}, & x \neq 0 \quad at\ x = 0 \\ 2 & , x = 0 \end{cases}$$

ii) $$f(x) = \begin{cases} \frac{x\left(3e^{1/x}+4\right)}{2-e^{1/x}}, & x \neq 0 \quad at\ x = 0 \\ 0 & , x = 0 \end{cases}$$

iii) $$f(x) = \begin{cases} \frac{e^x-1}{\log(1+2x)}, & x \neq 0 \quad at\ x = 0 \\ 7 & , x = 0 \end{cases}$$

iv) $$f(x) = \begin{cases} |x-a|\sin\left(\frac{1}{x-a}\right), & x \neq a \quad at\ x = a \\ 0 & , x = a \end{cases}$$

v) $$f(x) = \begin{cases} \frac{\sin 3x}{\tan 2x} & , x < 0 \\ \frac{3}{2} & , x = 0 \quad at\ x = 0 \\ \frac{\log(1+3x)}{e^{2x}-1}, & x > 0 \end{cases}$$

vi) $f(x) = |x-1| + |x+1|$ at $x=-1, 1$.

5) Find the value of a and b for each of the following functions are continuous at the indicated point(s).

i) $$f(x)=\begin{cases} x+a\sqrt{2}\sin x & , \ 0\le x<\pi/4 \\ 2x\cot x+b & , \ \pi/4\le x<\pi/2 \ \ at\ \pi/4\ and\ \pi/2 \\ a\cos 2x-b\sin x, & \pi/2\le x\le\pi \end{cases}$$

ii) $$f(x)=\begin{cases} \dfrac{x-4}{|x-4|}+a, & x<4 \\ a+b & , \ x=4, \ at\ x=4 \\ \dfrac{x-4}{|x-4|}+b, & x>4 \end{cases}$$

6) Show that $f(x)=|x-2|+|x-3|$ is not differentiable at $x=2$.

7) A function $f: R\to R$ satisfies that equation $f(x+y)=f(x).f(y)$ for all $x,y\in R$, $f(x)\neq 0$. Suppose that the function $f(x)$ is differentiable at $x=0$ and $f'(0)=2$. Prove $f'(x)=2f(x)$.

8) Find the values of a and b so that the function $f(x)=\begin{cases} x^2+3x+a, & x\le 1 \\ bx+2, & x>1 \end{cases}$ is differentiable at each $x\in R$.

9) Find the value of a so that the function $f(x)=\begin{cases} \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x>0 \\ a & , \ x=0 \\ \dfrac{1-\cos 4x}{x^2} & x<0 \end{cases}$ is continuous at $x=0$.

10) Find $\dfrac{dy}{dx}$, when

i) $y=\log\sqrt{\dfrac{1+\sin x}{1-\sin x}}$

ii) $y=\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right), \ -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

iii) $y=\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right), \ -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

iv) $y=\cos^{-1}\left\{\dfrac{2x-3\sqrt{1-x^2}}{\sqrt{13}}\right\}$

v) $xy=\sin(x+y)$

vi) $\tan(x+y) + \tan(x-y) = 1$

vii) $y = \log_{\sin x} \sec x + 10^{x^2}$

viii) $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

ix) $(\cos x)^y = (\cos y)^x$

x) $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ *and* $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

xi) $x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t - e^{-t}}{2}$

11) Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

12) If $y = \sqrt{\frac{1-x}{1+x}}$, prove that $(1-x^2)\frac{dy}{dx} + y = 0$

13) If $y = \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$, show that $\frac{dy}{dx}$ is independent of x.

14) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

15) If $y = e^x \sin x$ there prove $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

16) If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

17) If $x = a\sin 2\theta(1+\cos 2\theta)$ and $y = b\cos 2\theta(1-\cos 2\theta)$, show that at $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{b}{a}$.

18) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$.

19) If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, prove that $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$

20) If $x = a(1-\cos\theta)$, $y = a(\theta + \sin\theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \pi/2$

21) If $y = \left\{x + \sqrt{x^2+1}\right\}^m$, show that $(x^2+1)y_2 + xy_1 - m^2y = 0$.

22) Verify Rolle's theorem for each of the following functions on the indicated intervals.
   i) $f(x) = (x-1)(x-2)^2$ on $[1, 2]$
   ii) $f(x) = \log(x^2+2) - \log 3$ on $[-1, 1]$
   iii) $f(x) = \sin^4 x + \cos^4 x$ on $[0, \pi/2]$
   iv) $f(x) = e^x \sin x$ on $[0, \pi]$

23) Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0,4]$, where the tangent is parallel to the x-axis.

24) Verify Mean value theorem for each of the following functions on the indicated interval.
   i) $f(x) = x^3 - 2x^2 - x + 3$ on $[0, 1]$.
   ii) $f(x) = \sqrt{25 - x^2}$ on $[-3, 4]$.
   iii) $f(x) = \sin x - \sin 2x - x$ on $[0, \pi]$

25) Find the points on the curve $y = x^3 - 3x$, where the tangent to the curve is parallel to the chord joining (1–2) and (2,2).

ANSWERS

Section-A

1) i) b ii) c iii) d iv) a v) b vi) d
   vii) c viii) c ix) c x) d xi) c xii) b
   xiii) a xiv) a xv) b

2) i) $K=1$ ii) 0 iii) $\frac{4}{5}$ iv) 0 v) $2x$ vi) $\frac{1}{x \log_e^a}$
   vii) odd viii) $\frac{3}{4t}$ ix) $\frac{\pi}{180}\cos x^0$ x) 0

Section-B

3) ii) $x = 1, 2$

   v) a) $3^{\sin^2 x}\log 3.\sin 2x$ b) $\frac{5^x}{x^5}\left(\log 5 - \frac{5}{x}\right)$ c) $\frac{1}{\sqrt{1+x^2}}$

   d) $\frac{-\tan x}{\log 10 \log(\cos x)}$ e) $-\frac{1}{2}$ f) $\frac{3}{1+9x^2} - \frac{2}{1+4x^2}$

g) 1 h) $x^x(1+\log x)\cos(x^x)$ i) $x^x(1+\log x)$

vi) -4

vii) $-ne^{-nx}$

Section-C

4) i) Dicontinuous ii) Continuous iii) Discontinuous iv) Continuous
v) Continuous vi) Continuous

5) i) $a=\frac{\pi}{6}, b=-\frac{\pi}{12}$ ii) $a=1, b=-1$

8) $a=3,\ b=5$

9) $a=8$

10) i) $\sec x$ ii) -1 iii) $\frac{-x}{\sqrt{1-x^4}}$ iv) $\frac{1}{2\sqrt{1-x^2}}$ v) $\frac{\cos(x+y)-y}{x-\cos(x+y)}$

vi) $\frac{\sec^2(x+y)+\sec^2(x-y)}{\sec^2(x-y)-\sec^2(x+y)}$ vii) $\frac{\tan x\log\sin x+\cot x\log\cos x}{(\log\sin x)^2}+2x\log 10.10^{x^2}$

viii) $(\tan x)^{\cot x}.\mathrm{cosec}^2 x(1-\log\tan x)+(\cot x)^{\tan x}.\sec^2 x(\log\cot x-1)$

ix) $\frac{\log\cos y+y\tan x}{\log\cos x+x\tan y}$ x) $e^{-2\theta}\cdot\frac{\theta^2-\theta^3+\theta+1}{\theta^3+\theta^2+\theta-1}$ xi) $\frac{x}{y}$

11) $\frac{1}{4}$

23) $(2,-4)$

25) $\left(\pm\sqrt{\frac{7}{3}},\quad \frac{2}{3}\sqrt{\frac{7}{3}}\right)$

CHAPTER- 6

APPLICATION OF DERIVATIVES

Important points and Results :

- **Derivative as rate of change :**

  In various fields of applied mathematics one has the quest to know the rate at which one variable is changing with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variable also.

  e.g. : A physican may want to know, how small changes in dosage can affect the body's response to a drug.

- If $y = f(x)$, then $\dfrac{dy}{dx}$ measures the rate of change of y with respect to x.

- $\left(\dfrac{dy}{dx}\right)_{x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$.

- If the displacement s of a particle moving in a straight line at time t is given by $s = f(t)$ then

  i) $v =$ velocity at time $t = \dfrac{ds}{dt}$.

  $a =$ Acceleration at time $t = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = v.\dfrac{dv}{ds}$

  ii) When a particle moving in a straight line comes to rest, we have $\dfrac{ds}{dt} = 0$ and $\dfrac{d^2s}{dt^2} = 0$.

  iii) When a particle moving in a straight line is instantaneously at rest, we have $\dfrac{ds}{dt} = 0$ but $\dfrac{d^2s}{dt^2} \neq 0$.

- **Differentials, Errors and Approximations :**

- **Differentials :**

  Let $y = f(x)$ be a function of x, and let Δx be a small change in x. Let Δy be the corresponding change in y. Then,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = f'(x) + \in, \ where \in \to 0 \ as \ \Delta x \to 0$$

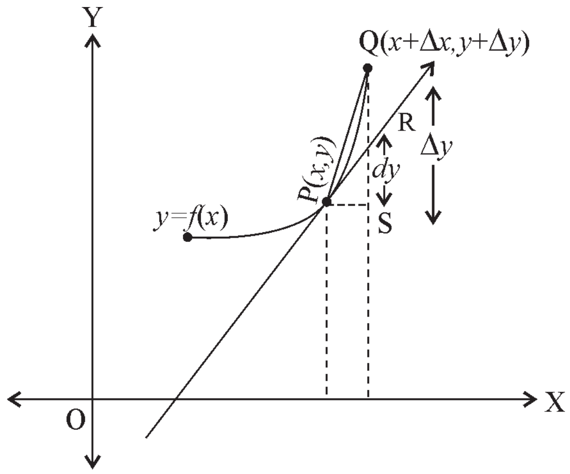
$$\Rightarrow \quad \Delta y = f'(x)\Delta x + \in \Delta x$$

$$\Rightarrow \quad \Delta y = f'(x)\Delta x \ \ (approx)$$

$$\Rightarrow \quad \Delta y = \frac{dy}{dx}\Delta x \ \ (approx)$$

- **Geometrical meaning of Differentials :**

Let us take a point P(x,y) on the curve $y=f(x)$. Let Q($x+\Delta x, y+\Delta y$) be a neighbouring point on the curve, where Δx denotes a small change in x and Δy is the corresponding change in y. From the adjacent figure, it is clear that $\frac{\Delta y}{\Delta x}$ is the slope of the secant PQ.



But, as $\Delta x \to 0$, $\frac{\Delta y}{\Delta x}$ approaches the limiting value $\frac{dy}{dx}$ (slope of the tangent at P). Therefore, when $\Delta x \to 0$, Δy (= QS) is approximately equal to dy (= RS).

We have,

$y + \Delta y = f(x+\Delta x)$

But, $\Delta y = \frac{dy}{dx}.\Delta x = f'(x)\Delta x$ *(approx)*

$\therefore \quad f(x+\Delta x) = y+f'(x)\Delta x$ *(approx)*

$$\Rightarrow \quad f(x+\Delta x) = y+\frac{dy}{dx}.\Delta x$$

Note : $\frac{dy}{dx}\Delta x$ is called differential of y and is denoted by dy.

- **Errors :** Let $y=f(x)$ be a given function of x. If Δx is an error in x, then the corresponding error Δy in y is given by $\Delta y = \frac{dy}{dx}\Delta x$

i) The error Δx in x and Δy in y are known as absolute errors.

ii) $\frac{\Delta x}{x}$ is called relative error in x.

iii) $\frac{\Delta x}{x}\times 100$ is called the percentage error in x.

- **Mean Value Theorems :**
- **Rolle's Theorem :**

If a function f defined on $[a,b]$ is

i) Continuous on $[a,b]$

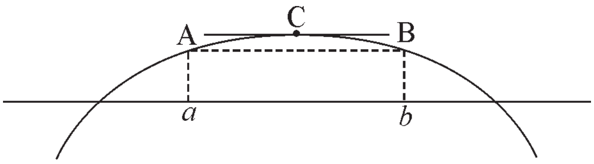
ii) derivable on (a,b) and

iii) $f(a) = f(b)$.

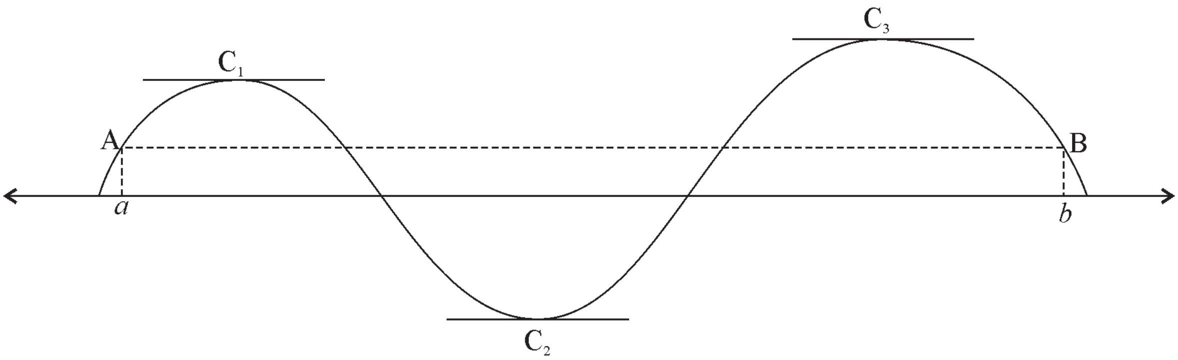
Then there exists at least one real number c between a and $b(a<c<b)$ such that $f'(c) = 0$.

- **Geometrical explanation of Rolle's Theorem :**

Let the curve $y = f(x)$, which is continuous on $[a,b]$ and derivable on (a,b) be drawn (as shown in figure).



$A(a, f(a)),\ B(b, f(b)), f(a)=f(b),\ C(c, f(c)),\ f'(c) = 0$



$C_1(c_1, f(c_1)), f'(c_1) = 0$

$C_2(c_2, f(c_2)), f'(c_2) = 0$

$C_3(c_3, f(c_3)), f'(c_3) = 0$

The theorem simply states that between two points with equal ordinates on the graph of $f(x)$, there exists at least one point where the tangent is parallel to x-axis.

- **Algebraic Interpretation of Rolle's Theorem :**

Between two zeroes a and b of $f(x)$ (i.e., between two roots a and b of $f(x) = 0$) there exists at least one zero of $f'(x)$.

- **Lagranges Mean Value Theorem (LMVT) :**

If a function f defined on $[a,b]$ is

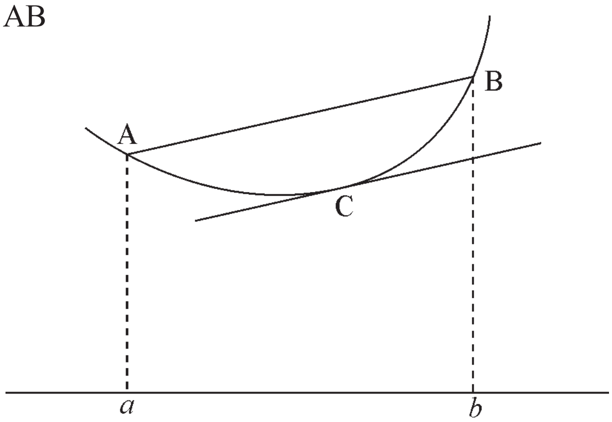
i) continuous on $[a,b]$ and

ii) derivable on (a,b)

then there exists at least one real number between a and $b(a<c<b)$ such that $\frac{f(b)-f(a)}{b-a}=f'(c)$.

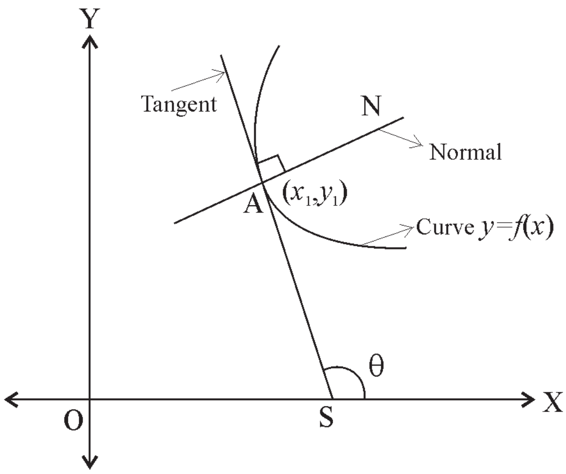
- **Geometrical Interpretation of LMVT :**
  The theorem simply states that between two points A and B of the graph $f(x)$ there exists at least one point where tangent is parallel to chord AB

$C(c, f(c))$, $f'(c) = \textit{Slope of }$ AB



- **Tangents and Normals**

- **Slope of the Tangent :**
  Let $y=f(x)$ be a curve and $A(x_1, y_1)$ be any point on the curve then $\left(\frac{dy}{dx}\right)_{A(x_1,y_1)}$ is known as the slope of the tangent to the curve $y=f(x)$ at point A. We denote the slope by $\tan\theta$ i.e., $\left(\frac{dy}{dx}\right)_{A(x_1,y_1)} = \tan\theta$, where θ is the angle made by the tangent in the positive direction of x-axis.



If $\theta = 0$, then $\left(\frac{dy}{dx}\right)_{A(x_1,y_1)} = \left(f'(x)\right)_{A(x_1,y_1)} = 0$.

If $\theta = 90^0$ i.e., tangent at A is parallel to y-axis.

- **Slope of the Normal :**
  Any line $\perp$ to the curve at $A(x_1,y_1)$ and also passes through $A(x_1,y_1)$ is called normal line.

Slope of normal at $A(x_1,y_1)$ is $= -\frac{1}{\text{Slope of tangent at } A(x_1, y_1)} = -\left(\frac{dy}{dx}\right)_{A(x_1,y_1)}$

- **Equation of the Tangent :**

  To find the equation of a line passing through a point A(x_1, y_1), we need the slope $\left(\frac{dy}{dx}\right)_{A(x_1, y_1)}$.

  Let us denote it by m, then equation of tangent is given by $y-y_1 = m(x-x_1)$ or

  $y-y_1=\left(\frac{dy}{dx}\right)_{A(x_1,y_1)}(x-x_1)$

- **Equation of Normal :**

  The line passing through A(x_1,y_1) and $\perp$ to the tangent to the curve is known as normal to the curve.

  As the slope of tangent at A(x_1,y_1) is $m=\left(\frac{dy}{dx}\right)_{A(x_1,y_1)}$. Therefore the slope of normal to be

  $-\frac{1}{\left(\frac{dy}{dx}\right)_{A(x_1,y_1)}}$ and the equation of normal to be given by

  $$y-y_1=-\frac{1}{\left(\frac{dy}{dx}\right)}(x-x_1)$$

  i.e., $m(y-y_1) = -(x-x_1)$
  or, $(x-x_1)+m(y-y_1) = 0$

- **Equation of Tangent and Normal in Parametric form :**

  Let $x = f(t)$, $y = g(t)$ be the equations of the curve in parametric form, then slope of the tangent at

  A(x_1,y_1) = $\frac{g'(t_1)}{f'(t_1)}$.

  In this case co-ordinates of A(x_1, y_1) are $(f(t_1), g(t_1))$ and equation of tangent and normal are given

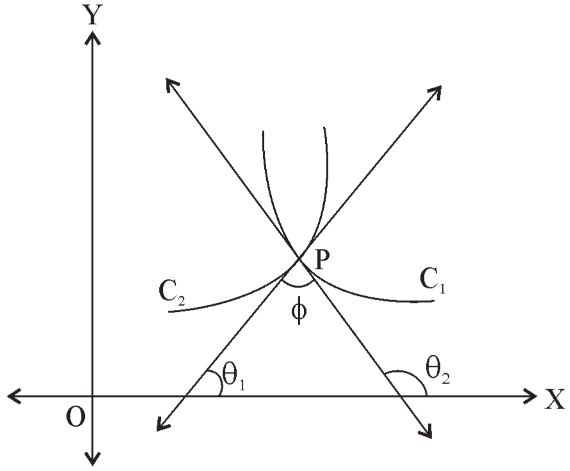
  by $y-g(t_1)=\frac{g'(t_1)}{f'(t_1)}(x-f(t_1))$ *and* $(y-g(t_1))g'(t_1)+(x-f(t_1))f'(t_1)=0$

- **Orthogonal Curves :**

  If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally and the curves are called orthogonal curves.

  If the curves are orthogonal, then $\phi = \pi/2$

$$\therefore\ m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_{c_1} \times \left(\frac{dy}{dx}\right)_{c_2} = -1$$



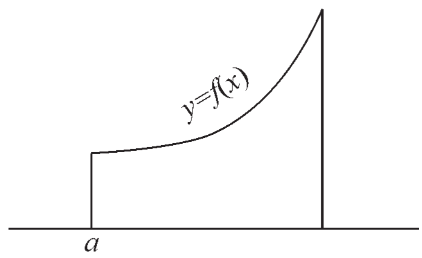
- **Monotonicity of a function :**

  A continous function $f(x)$, in an open interval $I = (a,b)$ is called monotonic if it satisfies any one of the following.

  Increasing if for all $x_1, x_2 \in I$,

  $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

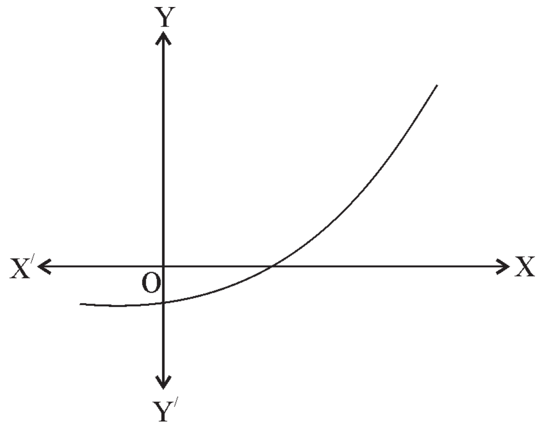
  Or, $x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$



- Strictly increasing if for all $x_1, x_2 \in I$,

  $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

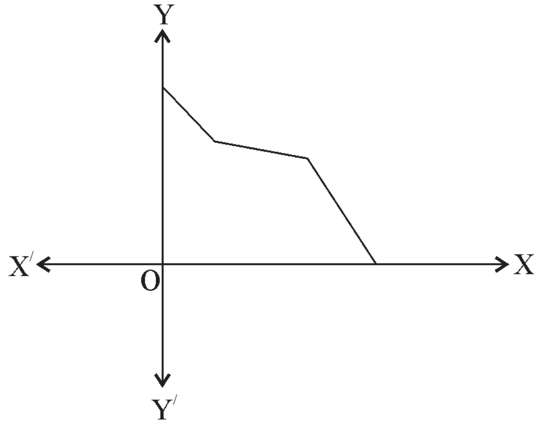
  Or, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$



- Decreasing if for all $x_1, x_2 \in I$,

  $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

  Or, $x_1 > x_2 \Rightarrow f(x_2) \ge f(x_1)$

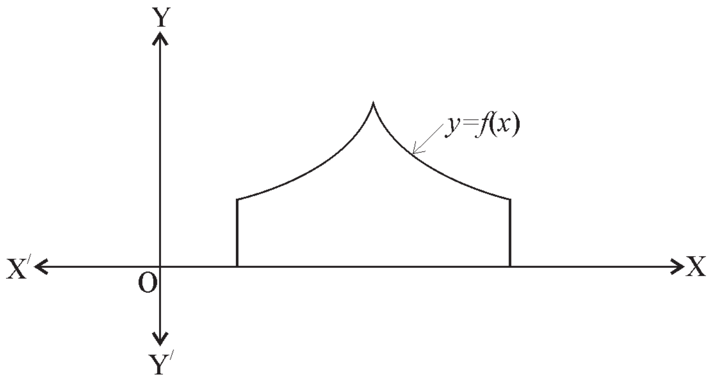


- Strictly decreasing if for all $x_1, x_2 \in$ I,

  $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

  Or, $x_1 > x_2 \Rightarrow f(x_2) > f(x_1)$

  Note : It is sometime possible that $y = f(x)$ is neither increasing nor decreasing in the given interval which can be seen by the diagram as follows :



- **Increasing and decreasing function at x_0 :**

  Let x_0 be a point in the domain of a real valued function f and there exists an open interval I=(x_0-h, x_0+h) containing x_0 such that

  (i) f is increasing at x_0, if $x_1 < x_2$ in I $\Rightarrow f(x_1) \le f(x_2)$

  (ii) f is strictly increasing at x_0, if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$

  (iii) f is decreasing at x_0, if $x_1 < x_2$ in I $\Rightarrow f(x_1) \ge f(x_2)$

  (iv) f is strictly decreasing at x_0, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

- **Test for increasing / decreasing / constant function :**

  Let f be a continuous function on $[a,b]$ and differentiable in (a,b), then

  (i) f is increasing on $[a,b]$ if $f'(x) > 0$ for each $x \in (a,b)$.

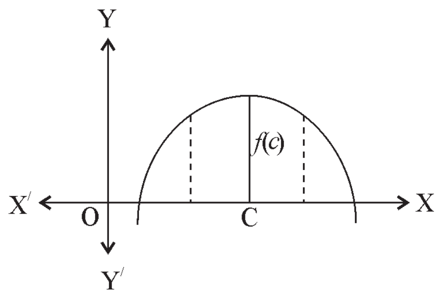
  (ii) f is decreasing on $[a,b]$ if $f'(x) < 0$ for each $x \in (a,b)$.

  (iii) f is constant on $[a,b]$ if $f'(x) = 0$ for each $x \in (a,b)$.

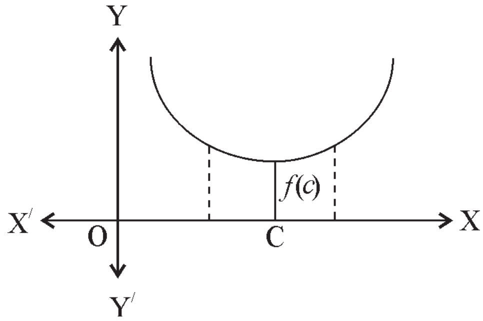
- **Maximum value, minimum value, extreme value :**

  Let f be a function defined in the interval I, then f has

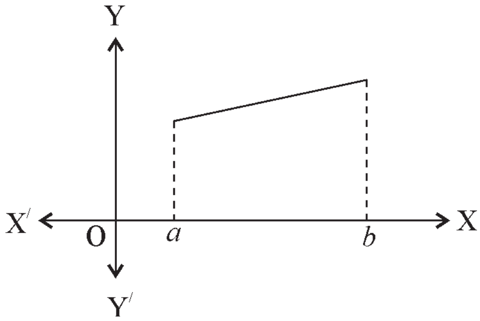
  (i) Maximum value if there exists a point c in I such that $f(c) \ge f(x)$, for all $x \in$ I. The point c is known as a point of maximum value in I.



(ii) Minimum value if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in$ I. The point c is called as a point of minimum value in I.



(iii) Extreme value if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value in I. The point c is said to be an extreme point.



- **Local maxima and minima :**

Let f be a real value function and c be an interior point in the domain of f, then

(i) Local maxima : c is a point of local maxima if there is an $h > 0$, such that $f(c) \geq f(x)$ for all $x \in (c-h, c+h)$.

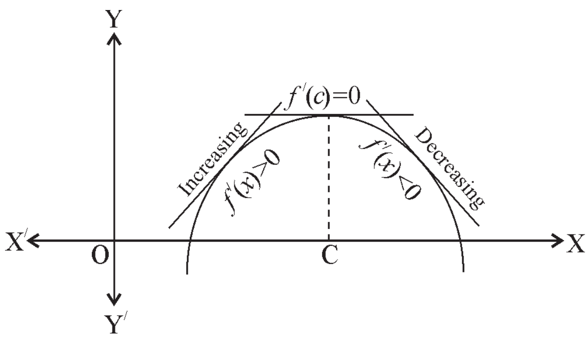
The value $f(c)$ is called local maximum value of f.

(ii) Local Minima : c is a point of local minima if there is an $h > 0$, such that $f(c) \leq f(x)$ for all $x \in (c-h, c+h)$.

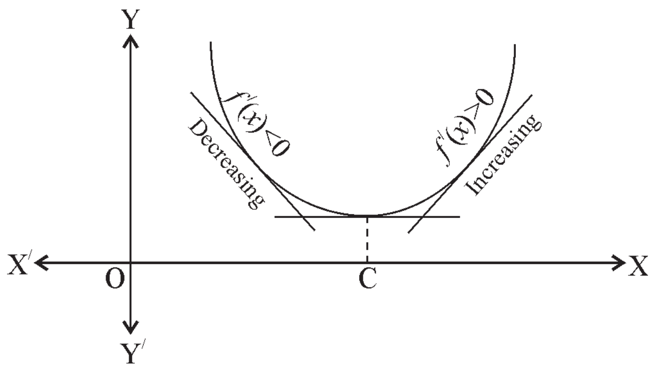
The value of $f(c)$ is known as the local minimum value of f.

- **Geometrical Interpretation :**

If $x = c$ is a point of local maxima of f, then f is increasing $(f'(x) > 0)$ in the interval $(c-h, c)$ and decreasing $(f'(x) < 0)$ in the interval $(c, c+h)$. This implies $f'(c) = 0$.

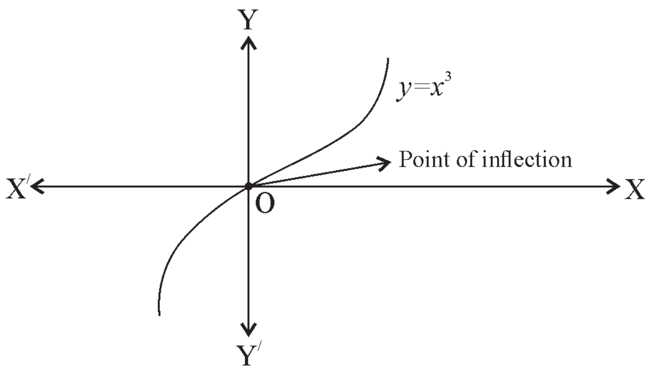


Again, if $x = c$ is a point of local minima of f, then f is decreasing $(f'(x) < 0)$ in the interval $(c-h, c)$ and increasing $(f'(x) > 0)$ in the interval $(c, c+h)$. This implies $f'(c) = 0$.



- **Point of inflection :**

  If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called point of inflection.



- **Second derivative test of maxima and minima :**

  Let f be a function defined on an interval I and $c \in$ I and f be differentiable at c. Then,

  (i) $x = c$ is a local maxima if $f'(c) = 0$ and $f''(c) < 0$.

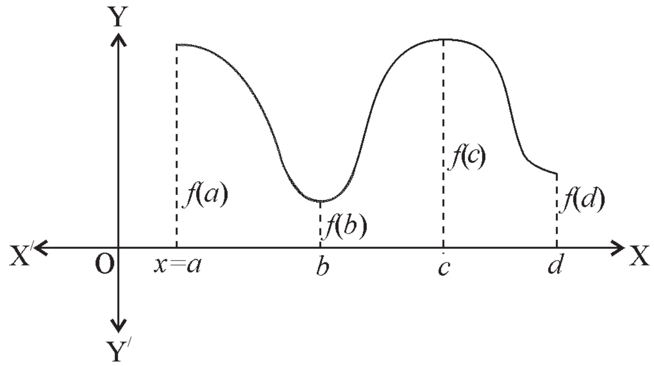
  (ii) $x = c$ is a local minima if $f'(c) = 0$ and $f''(c) > 0$.

  (iii) If $f'(c) = 0$ and $f''(c) = 0$, test fails, then we apply first derivative test as x increasing through c. $x = c$ is a point of inflection.

- **Maximum and minimum values in a closed interval :**

  Consider the function $f(x) = x+6, x \in [0,1]$. Here, $f'(c) \neq 0$. It has neither maxima nor minima. But $f(0) = 6$. This is the absolute minimum or global minimum or least value.

  Further $f(1) = 7$. This is the absolute maximum or global maximum or greatest value.



- **To find absolute maximum value or absolute minimum value :**
  (i) Find all the critical points viz is where $f'(x) = 0$ or f is not differentiable.
  (ii) Consider the end points also.
  (iii) Calculate the functional values at all the points found (i) & (ii).
  (iv) Identify the maximum and minimum values out of the values calculated in (iii).
  These are absolute maximum and absolute minimum values.

- **Useful formulae of Mensuration to Remember :**
  1. Volume of a cuboid = lbh
  2. Surface area of a cuboid = $2(lb+bh+hl)$
  3. Volume of a cube = a^3
  4. Surface area of a cube = $6a^2$
  5. Volume of a cone = $\frac{1}{3}\pi r^2 h$
  6. Curves surface are of a cone = πrl (l = slant height)
  7. Total surface area of cone = $\pi r(r+l)$
  8. Total surface area of a cylinder = $2\pi r(r+h)$
  9. Volume of a cylinder = $\pi r^2 h$
  10. Volume of a sphere = $\frac{4}{3}\pi r^3$
  11. Surface area of a sphere = $4\pi r^2$
  12. Volume of a hemisphere = $\frac{2}{3}\pi r^3$
  13. Surface area of a hemisphere = $3\pi r^2$
  14. Volume of a prism = (area of the base) × (height)
  15. Lateral surface area of a prism = (Perimiter of the base) × (height)
  16. Total surface area of a prism = (lateral surface area) +2(area of the base)
  17. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × height
  18. Curved surface area of a pyramid = $\frac{1}{2}$ (Perimeter of the base) × (slant height)

Exercise - 6

Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) Multiple choice type questions :

i) If $S = t^3 - 4t^2 + 5$ describes the motion of a particle, then its velocity when the aceleration vanishes is

a) $\frac{16}{9}$ unit/sec b) $\frac{-32}{3}$ unit/sec c) $\frac{4}{3}$ unit/sec d) $\frac{-16}{3}$ unit/sec

ii) Each side of an equilateral triangle is increasing at the rate of 8cm/hr. The rate of increase of its area when side is 2cm, is

a) $8\sqrt{3}cm^2 / hr$ b) $4\sqrt{3}\, cm^2 / hr$ c) $\frac{\sqrt{3}}{8} cm^2 / hr$ d) none of these

iii) A man of height 6ft walks at a uniform speed of 9ft/sec from a lamp fixed at 15ft height. The length of his shadow is increasing at the rate of

a) 15 ft/sec b) 9 ft/sec c) 6ft/sec d) none of these

iv) If the rate of change of volume of a sphere is equal to the rate of change of its radius, then its radius is equal to

a) 1 unit b) $\sqrt{2\pi}$ unit c) $\frac{1}{\sqrt{2\pi}}$ unit d) $\frac{1}{2\sqrt{\pi}}$ unit

v) For what values of x is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x?

a) $-3, -\frac{1}{3}$ b) $-3, \frac{1}{3}$ c) $3, -\frac{1}{3}$ d) $3, \frac{1}{3}$

vi) The coordinates of the point on the ellipse $16x^2 + 9y^2 = 400$ where the ordinate decreases at the same rate at which the abscissa increases are

a) $\left(3, {}^{16}\!/_{3}\right)$ b) $\left(-3, {}^{16}\!/_{3}\right)$ c) $\left(3, -{}^{16}\!/_{3}\right)$ d) (3, –3)

vii) A cylindrical vessel of radius 0.5m is filled with oil at the rate of 0.25π m^3/minute. The rate 0.25π at which the surface of the oil is rising, is

a) 1m/minute b) 2m/minute c) 5m/minute d) 1.25m/minute

viii) The attitude of a cone is 20 cm and its semi-vertical angle is 30^0. If the semi-vertical angle is increasing at the rate of 2^0 per seccond, then the radious of the base is increasing at the rate of

a) 30 cm/sec b) $\frac{160}{3}$ cm/sec c) 10 m/sec d) 160 cm/sec.

ix) If there is an error of 2% in measuring the length of a simple pendulum, then percentage error in its period is

a) 1% b) 2% c) 3% d) 4%

x) If $\log_e^4 = 1.3868$, then $\log_e^{4.01} =$

a) 1.3968 b) 1.3898 c) 1.3893 d) none of these

xi) If $y = x^n$, then the ratio of relative errors in y and x is

a) 1:1 b) 2:1 c) 1:n d) n:1

xii) If there is an error of x% in measuring the edge of a cube, then percentage error in its surface is

a) $2x$% b) $\frac{x}{2}$% c) $3x$% d) none of these

xiii) A sphere of radius 100 mm shrinks to radius 98 mm, then the approximate decrease in its volume is

a) 12000π mm^3 b) 800π mm^3 c) 80000π mm^3 d) 120π mm^3

xiv) If the ratio of base radius and height of a cone is 1:2 and percentage error in radius is λ%, then the error in its volume is

a) λ% b) 2λ% c) 3λ% d) none of these

xv) If $4a+2b+c = 0$, then the equation $3ax^2+2bx+c = 0$ has at least one real root lying in the interval

a) (0,1) b) (1,2) c) (0,2) d) none of these

xvi) When the tangent to the curve $y = x\log x$ is parallel to the chord joining the points (1,0) and (e,e), the value of x is

a) $e^{1/(1-e)}$ b) $e^{(e-1)(2e-1)}$ c) $e^{\frac{2e-1}{e-1}}$ d) $\frac{e-1}{e}$

xvii) The curves $y = ae^x$ and $y = be^{-x}$ cut orthogonally, if

a) $a=b$ b) $a = -b$ c) $ab = 1$ d) $ab = 2$

xviii) The point on the curve $y^2 = x$ where tangent makes 45^0 angle with x-axis is

a) $\left(1/2, 1/4\right)$ b) $\left(1/4, 1/2\right)$ c) (4, 2) d) (1, 1)

xix) Any tangent to the curve $y = 2x^7+3x+5$

a) is parallel to x-axis b) is parallel to y-axis

c) makes an acute angle with x-axis d) makes an obtuse angle with x-axis

xx) The point on the curve $y = 6x-x^2$ at which the tangent to the curve is inclined at $\pi/4$ to the line $x+y = 0$ is

a) (–3, –27) b) (3, 9) c) $\left(7/2, 35/4\right)$ d) (0, 0)

xxi) The equation of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at the point $\theta = \pi/4$ is

a) $x = 0$ b) $y = 0$ c) $x = y$ d) $x+y = a$

xxii) Slope of the normal at the point 't' on the curve $x = \frac{1}{t}$, $y = t$ is

a) $\frac{1}{t}$ b) $\frac{1}{t^2}$ c) t d) $-t$

xxiii) The equation of tangent at those points where the curve $y=x^2-3x+2$ meets x-axis are

a) $x-y+2 = 0 = x-y-1$ b) $x+y-1 = 0 = x-y-2$

c) $x-y-1 = 0 = x-y$ d) $x-y = 0 = x+y$

xxiv) In the interval (1, 2), the function $f(x)=2|x-1|+3|x-2|$ is

a) increasing b) decreasing c) constant d) none of these

xxv) The function $f(x)=\frac{x}{1+|x|}$ is

a) strictly increasing b) strictly decreasing

c) neither increasing nor decreasing d) none of these

xxvi) If the function $f(x)=x^2-Kx+5$ is increasing on [2, 4], then

a) $K\in(2,\infty)$ b) $K\in(-\infty, 2)$ c) $K\in(4,\infty)$ d) $K\in(-\infty, 4)$

xxvii) $f(x)=2x-\tan^{-1}x-\log\left(x+\sqrt{x^2+1}\right)$ is monotonically increasing when

a) $x>0$ b) $x<0$ c) $x\in R$ d) $x\in R-\{0\}$

xxviii) The function $f(x)=x^x$ decreases on the interval

a) (0, e) b) (0, 1) c) $\left(0, 1/e\right)$ d) none of these

xxix) The least and greatest values of $f(x)=x^3-6x^2+9x$ in [0,6], are

a) 3, 4 b) 0,6 c) 0,3 d) 3, 6

xxx) Let $f(x)=(x-a)^2+(x-b)^2+(x-c)^2$. Then, $f(x)$ has a minimum at $x=$

a) $\frac{a+b+c}{3}$ b) $\sqrt[3]{abc}$ c) $\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$ d) none of these.

2. Very short answer type questions :

i) If a particle moves in a straight line such that the distance travelled in time t is given by $s=t^3-6t^2+9t+8$. Find the initial velocity of the particle.

ii) The side of a square is increasing at the rate of 0.1cm/sec. Find the rate of increase of its perimeter.

iii) If $y=\log_e^x$, then find Δy when $x=3$ and $\Delta x=0.03$.

iv) A piece of ice is in the form of a cube melts so that the percentage error in the edge of cube is a, then find the percentage error in its volume.

v) Find the slope of the tangent to the curve $x=t^2+3t-8$, $y=2t^2-2t-5$ at $t=2$.

vi) Write the value of $\frac{dy}{dx}$, if the normal to the curve $y=f(x)$ at (x,y) is parallel to y-axis.

vii) If the tangent to a curve at a point (x,y) is equally inclined to the co-ordinate axes, then write the value of $\frac{dy}{dx}$.

viii) Write the equation of the normal to the curve $y = x + \sin x \cos x$ at $x = \frac{\pi}{2}$.

ix) Write the equation of the tangent to the curve $y = x^2 - x + 2$ at the point where it crosses the y-axis.

x) Write the set of values of a for which $f(x) = \cos x + a^2 x + b$ is strictly increasing on R.

xi) State whether $f(x) = \tan x - x$ is increasing or decreasing in its domain.

xii) Find the set of values of 'b' for which $f(x) = b\,(x + \cos x) + 4$ is decreasing on R.

xiii) Write the maximum value of $f(x) = x^{1/x}$.

xiv) Find the least value of $f(x) = ax + \frac{b}{x}$, where $a > 0$, $b > 0$ and $x > 0$.

xv) Write the point where $f(x) = x \log_e^x$ attains minimum value.

Section - B

3] Short answer type questions : [Each question carries 3 marks]

i) Find the intervals in which the following functions are increasing or decreasing.

a) $f(x) = x^4 - 4x^3 + 4x^2 + 15$

b) $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$

c) $f(x) = (x + 2)\, e^{-x}$

d) $f(x) = \frac{x}{\log x}$

e) $f(x) = x^x$

f) $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$, $0 \le x \le 2\pi$

g) $f(x) = (x+1)^3 (x-3)^3$

h) $f(x) = \frac{4x^2 + 1}{x}$

ii) Prove that the function $f(x) = \frac{x-2}{x+1}$ is increasing for all X ε R, except $x = -1$.

iii) Prove that $f(\theta)=\frac{4\sin\theta}{2+\cos\theta}-\theta$ is an increasing function of θ in $\left[0,\frac{\pi}{2}\right]$

iv) Show that the function $f(x)=x^{100}+\sin x-1$ is neither increasing nor decreasing on $(-1,1)$.

v) Find the least value of a such that the function $x^2+3ax+5$ is increasing on $[1,2]$.

vi) Find the values 'a' for which the function $f(x)=(a+2)x^3-3ax^2+9ax-1$ decreases for all real x.

vii) Show that $f(x)=\log\sin x$ is increasing on $\left(0,\pi/2\right)$ and decreasing on $\left(\pi/2,\pi\right)$.

viii) show that $f(x)=\tan^{-1}(\sin x+\cos x)$ is a decreasing function on the intervel $\left(\pi/4,\pi/2\right)$.

ix) Find the intervals in which $f(x)=\log(1+x)-\frac{x}{1+x}$ is increasing or decreasing.

x) Find the intervals in which $f(x)=(x+2)e^{-x}$ is increasing or decreasing.

xi) Prove that the function $f(x)=x-[x]$ is increasing in $(0,1)$.

xii) Find the points of local maxima or local minima, if any, of the following functions. Also, find the local maximum or local minimum values :

a) $f(x)=\sin^4 x+\cos^4 x,\ 0<x<\frac{\pi}{2}$

b) $f(x)=2\sin x-x,\ -\frac{\pi}{2}\le x\le\frac{\pi}{2}$

c) $f(x)=\cos x,\ 0<x<\pi$

d) $f(x)=(x-1)^3(x+1)^2$

xiii) Find the slope of the normal to the curve $x=1-a\sin\theta,\ y=b\cos^2\theta$ at $\theta=\frac{\pi}{2}$.

xiv) Find points on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which the tangents are parallel to the y-axis.

xv) Find the values of a and b if the slope of the tangent to the curve $xy+ax+by=2$ at $(1,1)$ is 2.

xvi) Find the point on the curve $y=x^2$ where the slope of the tangent is equal to the x-coordinate of the point.

xvii) For what value of 'a', $3x+4y=1$ is a tangent of $y^2=4ax$?

Section-C

4. **Long answer type questions :** (4 or 6 marks each)

i) The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3cm/sec. How fast is the area decreasing when the two equal sides are equal to the base ?

ii) The volume of metal in a hollow sphere is constant. If the inner radius is increasing at the rate of 1 cm/sec, find the rate of increase of the outer radius when the radii are 4cm and 8cm respectively.

iii) The time of T of a complete oscillation of a simple pendulum of length l is given by the equation $T = 2\pi\sqrt{l/g}$, where g is constant. What is the percentage error in T when l is increased by 1%?

iv) If in a triangle ABC, the side c and the angle C remain constant, using differentials show that –

$$\frac{da}{\cos A}+\frac{db}{\cos B}=0$$

v) Using differentials, find the approximate values of the following :

a) $\tan 46^0$, it is given that $1^0 = 0.01745$ radian.

b) $\log_e^{4.04}$, it is given that $\log_{10}^{4} = 0.6021$ and $\log_{10}^{e} = 0.4343$.

c) $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$

d) $\sin\left(\frac{22}{14}\right)$

e) $\frac{1}{\sqrt{25.1}}$

vi) Find the equations of the tangent and the normal to the following curves at the indicated points :

a) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \; at \; (a\sec\theta, b\tan\theta)$

b) $x^{2/3}+y^{2/3}=2 \; at \; (1,1)$

c) $y^2 = 4ax \;\; at \; \left(a/m^2, 2a/m\right)$

d) $x=\frac{2at^2}{1+t^2},\; y=\frac{2at^3}{1+t^2} \; at \; t=\frac{1}{2}$

e) $x = a\sin^3 t,\; y = b\cos^3 t$ at t.

vii) Find the points on the curve $4x^2 + 9y^2 = 1$, where the tangents are perpendicular to the line $2y + x = 0$.

viii) Show that the normal at any point θ to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = (\sin\theta - \theta\cos\theta)$ is at a constant distance from the origin.

ix) Find the angle between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at their point of intersection other than the origin.

x) Show that the condition that the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ should intersect orthogonally is that $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$.

xi) Show that the curves $4x = y^2$ and $4xy = K$ cut at right angles, if $K^2 = 512$.

xii) If the sum of the lengths of the hypotenues and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

xiii) Find the equation of tangents to the circle $x^2 + y^2 = 3$, Which makes 60^o with the positive direction of x axis.

xiv) Find the equation of normal to the hyperbola $x^2 - y^2 = 16$ at $(4 \sec\theta, 4 \tan\theta)$. Hence show that $2x + 4y = 9$ is a normal to the hyperbola.

xv) Find the equation of normal to the parabola $y^2 = 12x$ at $(3t^2, 6t)$. Hence find the equation of normal to the parabola which makes 135^o with positive direction of x axis.

xvi) If $lm + my + n = 0$ is a tangent of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove $a^2l^2 + b^2m^2 = n^2$.

xvii) If $lx + my = 1$ is a normal to the parabola $y^2 = 4ax$ then prove that $al^3 + 2alm^2 = m^2$.

xviii) Prove that sum of intercepts on the axes made by a tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at any point on it is constant.

xix) Prove that an isosceles triangle that can be inscribed in a given circle will have the greatest area if it is an equilateral triangle.

xx) Length of hypoteneous of a right angled triangle is 5 cm. Find the greatest value of its area.

xxi) A rectangle is inscribed in a semi-circle of radius 'r' in such a way that one side of rectangle coincides with the diameter of semi-circle. If the area of rectangle is maximum find its length and breadth in terms of 'r'.

xxii) Find the greatest volume of cylinder that can be inscribed in a sphere of raidus $5\sqrt{3}$ cm.

xxiii) An open box with square base is made by a board of area c^2 sq. units. Prove that maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

xxiv) Find the point on $x^2 = 2y$ which is nearest to $(0, 3)$.

xxv) If the function $f(x) = 4x^3 + ax^2 + bx + 2$ has the extreme value at $(2, -2)$ then find the value 'a' or 'b'. Also show at that point the function has maximum value.

ANSWERS

Section-A

1)	i) d	ii) a	iii) c	iv) d	v) d	vi) a
	vii) a	viii) b	ix) a	x) c	xi) d	xii) a
	xiii) c	xiv) c	xv) c	xvi) a	xvii) c	xviii) b
	xix) c	xx) b	xxi) c	xxii) b	xxiii) b	xxiv) b
	xxv) a	xxvi) d	xxvii) c	xxviii) c	xxix) a	xxx) a

2) i) 9 units/unit time ii) 0.4cm/sec iii) 0.01 iv) $3a$ v) $\frac{6}{7}$ vi) 0

vii) ± 1 viii) $2x = \pi$ ix) $x+y-2=0$ x) $a \in (-\infty, -1] \cup [1, \infty)$ xi) increasing

xii) $b \in (-\infty, 0)$ xiii) $e^{\frac{1}{e}}$ xiv) $2\sqrt{ab}$ xv) $\left(\frac{1}{e}, -\frac{1}{e}\right)$

Section-B

3) i)

	Increasing	Decreasing
a)	$[0, 1] \cup [2\ \infty)$	$(-\infty, -2] \cup [1, 2]$
b)	$(-2, 1) \cup (3, \infty)$	$(-\infty, -2) \cup (1, 3)$
c)	$[-\infty, -1]$	$[-1, \infty)$
d)	(e, ∞)	$(0, e) - \{1\}$
e)	$\left(\frac{1}{e}, \infty\right)$	$\left(0, \frac{1}{e}\right)$
f)	$\left(0, \frac{\pi}{2}\right)$ $\left(\frac{3\pi}{2}, 2\pi\right)$	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
g)	$(1, \infty)$	$(-\infty, 1)$
h)	$\left(-\infty, \frac{1}{2}\right)$ $\left(\frac{1}{2}, -\infty\right)$	$\left(-\frac{1}{2}, 0\right)$ $\left(0, \frac{1}{2}\right)$

v) -1 vi) $a \in (-\infty, -3)$ ix) inceasing on $(0, \infty)$ decreasing on $(-1, 0)$

x) Increasing on $(-\infty, -1)$, decreasing on $(-1, \infty)$

xii) a) $x = \frac{\pi}{4}$ is a point of local minimum and local minimum value $= \frac{1}{2}$

b) $x = \frac{\pi}{3}$, point of local maximum, local maximum value $\sqrt{3} - \frac{\pi}{3}$

$x = -\frac{\pi}{3}$, point of local minimum, local min value $= -\sqrt{3} + \frac{\pi}{3}$

c) None in the interval $(0, \pi)$

d) $x = -1$, is a point of local max., local max value=0

$x=-\frac{1}{5}$, is a point of local min., local min value $=-\frac{3456}{3125}$

xiii) $-\frac{a}{2b}$ xiv) $(\pm 3, 0)$ xv) $a=5, b=-4$ xvi) $(0, 0)$ xvii) $-\frac{3}{16}$

Section-C

4. i) $\sqrt{3}bcm^2/\sec$ ii) $\frac{1}{4}cm/\sec$ iii) $\frac{1}{2}\%$

v) a) 1.03490 b) 1.396368 c) –34.99 d) 1 e) 0.198

vi) Tangent Normal

a) $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$ $ax\cos\theta+by\cot\theta=a^2+b^2$

b) $x+y-2=0$ $y-x=0$

c) $m^2x-my+a=0$ $m^2x+m^3y-2am^2-a=0$

d) $13x-16y-2a=0$ $16+13y-19a=0$

e) $b\,x\cos t+a\,y\sin t=ab\sin t\cos t$ $a\,x\sin t-b\,y\cos t=a^2\sin^4 t-t^2\cos^4 t$.

vii) $\left(\frac{3}{2\sqrt{10}},\frac{-1}{3\sqrt{10}}\right)$ *and* $\left(\frac{-3}{2\sqrt{10}},\frac{1}{3\sqrt{10}}\right)$

ix) $\tan^{-1}\left\{\frac{3(ab)^{1/3}}{2\left(a^{2/3}+b^{2/3}\right)}\right\}$

xiii) $y=\sqrt{3}x\pm 2\sqrt{3}$

xiv) $x\cos\theta+y\cot\theta=8$

xv) $tx+y=6t+3t^3$ and $x+y=9$.

xx) $\frac{25}{4}$ sq. cm.

xxi) Length= $r\sqrt{2}$ unit, breadth $=\frac{r}{\sqrt{2}}$ unit.

xxii) 500π cm^3.

xxiv) (2, 2) ও (–2, 2) ।

xxv) a = –15, b = 12 ।

CHAPTER- 7

INTEGRALS

Important points and Results :

- Integration is the inverse process of defferentiation. Let $\frac{d}{dx}F(x)=f(x)$. Then we write $\int f(x)dx=F(x)+C$. These integrals are called indefinite integrals or general integrals, C is called constant of integration. All these integrals differ by a constant.

- Geometrically, an indefinite integral is collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards along y-axis.

- **Some properties of indefinite integrals :**

  i) The process of differentiation and integration are inverse of each other, i.e., $\frac{d}{dx}\int f(x)dx=f(x)$ and $\int f'(x)dx=f(x)+c$, where c is any arbitary cosntant.

  ii) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent. So if f and g are two functions such that $\frac{d}{dx}\int f(x)dx=\frac{d}{dx}\int g(x)dx$, then $\int f(x)dx$ and $\int g(x)dx$ are equivalent.

  iii) The integral of the sum or the difference of two functions is equal to the sum or difference of their integrals. i.e,

  $$\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$$

  iv) A constant factor may be written either before or after the integral sign, i.e.,

  $\int af(x)dx=a\int f(x)dx$, where '$a$' is a constant.

  v) Properties (iii) and (iv) can be generalised to a finite number of functions $f_1, f_2, \ldots\ldots\ldots, f_n$ and real numbers $k_1, k_2, \ldots\ldots, k_n$ giving

  $$\int[K_1f_1(x)\pm K_2f_2(x)\pm\cdots\pm K_nf_n(x)]dx=K_1\int f_1(x)dx\pm K_2\int f_2(x)dx\pm\cdots\pm K_n\int f_n(x)dx$$

- **Some standard integrals :**

  i) $\int x^n dx=\frac{x^{n+1}}{n+1}+c,\quad n\neq-1,\quad particularly,\ \int dx=x+c$

  ii) $\int\frac{1}{x}dx=\log|x|+c$

iii) $\int 0.dx = c$

iv) $\int e^x\, dx = e^x + c$

v) $\int a^x dx = \dfrac{a^x}{\log a} + c$

vi) $\int \cos x dx = \sin x + c$

vii) $\int \sin x dx = -\cos x + c$

viii) $\int \sec^2 x dx = \tan x + c$

ix) $\int \text{cosec}^2 x dx = -\cot x + c$

x) $\int \sec x \tan x dx = \sec x + c$

xi) $\int \text{cosec}\, x \cot x dx = -\text{cosec}\ x + c$

xii) $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c = -\cos^{-1} x + c$

xiii) $\int \dfrac{dx}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$

xiv) $\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\text{cosec}^{-1} x + c$

- **Methods of integration :**
- **Method of Transformation –**

  When the integrand is a trigonometric function, we transform the given function into standard integrals or their algebraic sum by using trigonometric formulae.

- **Method of substitution –**

  By suitable substitution, the variable x in $\int f(x)\, dx$ is changed into another variable t so that the integrand $f(x)$ is changed into $F(t)$ which is some standard integral or algebriac sum of standard integrals. There is no general rule for finding a proper substitution. Using substitution technique, we obtain the following standard integrals.

i) $\int \tan x\, dx = -\log|\cos x| + c = \log|\sec x| + c$

ii) $\int \cot x\, dx = \log|\sin x| + c$

iii) $\int \sec x\, dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\right| + c$

iv) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c = \log\left|\tan\frac{x}{2}\right| + c$

- **Integrals of some special functions :**

i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

iii) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$

- **Integration by parts :**

For given functions f_1 and f_2, we have $\int f_1(x).f_2(x)dx = f_1(x)\int f_2(x)dx - \int\left[\frac{d}{dx}f_1(x).\int f_2(x)dx\right]dx$

i.e., the integral of the product of two functions = First function × integral of the second function – integral of {differential coefficient of the first function × integral of the second function}. Care must taken in choosing the first function and the second function. Obviously, we must take that function as the second function whose integral is well known to us.

If two funcions are of different types, then consider the first function which comes first in the word "ILATE" where

I : Inverse trigonometric function.
L : Logarithmic function.
A : Algebraic function.
T : Trigonometric function.
E : Exponential function.

$\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c$

- **Some special types of integrals :**

i) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

ii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

iii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

iv) Integrals of the types $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ or $\int \sqrt{ax^2+bx+c}\,dx$ can be transformed into standard form by expressing

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

v) Integrals of the types $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ or $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ can be transformed into standard form by expressing $px+q = A\frac{d}{dx}(ax^2+bx+c)+B = A(2ax+b)+B$, where A and B are determined by comparing coefficients on both sides.

- **Integration by partial fractions :**

Integrals of the type $\int \frac{p(x)}{g(x)}$ can be integrated by solving the integrand into partial fractions. We proceed as follows :

Check degree of $p(x)$ < degree of $g(x)$, otherwise divide $p(x)$ by $g(x)$ till its degree is less i.e., put in the form $\frac{p(x)}{g(x)} = r(x)+\frac{f(x)}{g(x)}$, where degree of $f(x)$ < degree of $g(x)$.

Now $\frac{f(x)}{g(x)}$ can be integrated by expressing $\frac{f(x)}{g(x)}$ as the sum of partial fractions of the following types.

i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a}+\frac{B}{x-b},\ a \neq b$

ii) $\frac{px+q}{(x-a)^2} = \frac{A}{x-a}+\frac{B}{(x-a)^2}$

iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a}+\frac{B}{(x-b)}+\frac{C}{x-c}$

iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$

v) $$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$$

Where x^2+bx+c cannot be factorised further.

- **Definite integral :**

The definite integral is denoted by $\int_a^b f(x)dx$, where a is the lower limit of the integral and b is the upper limit of the integral. The definite integral is evaluated in the following two ways.

i) The definite integral as the limit of the sum.

ii) $\int_a^b f(x)dx = F(b)-F(a)$, if $F(x)$ is an antiderivative of $f(x)$.

- **The definite integral as the limit of the sum :**

The definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y=f(x)$, the ordinates $x=a, x=b$ and the x-axis and given by $\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+.......+f\{a+(n-1)h\}\right]$ where

$h=\frac{b-a}{n}\to 0$ as $n\to\propto$.

- **Fundamental Theorem of calculus :**

i) Area function : The function $A(x)$ denotes the area function and is given by $A(x)=\int_a^x f(x)dx$.

ii) First Fundamental Theorem of integral calculus :
Let f be a continuous function on the closed interval $[a,b]$ and let $A(x)$ be the area function. Then $A'(x)=f(x)$, for all $x\in[a,b]$.

iii) Second Fundamental Theorem of integral calculus :
Let f be continuous function defined on the closed interval $[a,b]$ and F be an antiderivative of f. Then $\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a)$.

- **Some properties Definite Integrals :**

i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

ii) $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

iii) $$\int_a^a f(x)dx = 0$$

iv) $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \ where \ a < c < b$$

v) $$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

vi) $$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

vii) $$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

viii) $$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if \ \ f(2a-x) = f(x) \\ 0 & , \ if \ \ f(2a-x) = -f(x) \end{cases}$$

ix) $$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if \ \ f(-x) = f(x), \ i.e. \ even \ \ function. \\ 0 & , \ if \ \ f(-x) = -f(x), \ i.e. \ odd \ \ function. \end{cases}$$

Exercise-7
Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) Multiple choice type questions :

i) The antiderivative of the function $f(x)=\left(1-\frac{1}{x^2}\right)a^{x+\frac{1}{x}}$, $a>0$ is

a) $\frac{a^{x+\frac{1}{x}}}{a}$ b) $\frac{a^{x+\frac{1}{x}}}{x}$ c) $\frac{a^{x+\frac{1}{x}}}{\log a}$ d) $a^{x+\frac{1}{x}}.\log a$

ii) $\int e^x(1-\cot x+\cot^2 x)dx$ =

a) $e^x\cot x+c$ b) $-e^x\cot x+c$ c) $e^x\operatorname{cosec} x+c$ d) $-e^x\operatorname{cosec} x+c$

iii) If $\int \frac{2^{1/x}}{x^2}dx = k2^{1/x}+c$, then k is equal to

a) $-\frac{1}{\log_e^2}$ b) $-\log_e^2$ c) -1 d) $\frac{1}{2}$

iv) If $\int \frac{\sin x}{\sin(x-\alpha)}dx = Ax+B\log\sin(x-\alpha)+C$, then the value of (A, B) is

a) $(-\sin\alpha, \cos\alpha)$ b) $(\cos\alpha, \sin\alpha)$ c) $(\sin\alpha, \cos\alpha)$ d) $(-\cos\alpha, \sin\alpha)$

v) $\int \frac{dx}{\sin^2 x\cos^2 x}$ is equal to

a) $\tan x+\cot x+c$ b) $(\tan x+\cot x)^2+c$ c) $\tan x-\cot x+c$ d) $(\tan x-\cot x)^2+c$

vi) $\int_{a+c}^{b+c} f(x)dx$ is equal to

a) $\int_a^b f(x)dx$ b) $\int_{a-c}^{b-c} f(x)dx$ c) $\int_a^b f(x-c)dx$ d) $\int_a^b f(x+c)dx$

vii) $\int \frac{d^2}{dx^2}(\tan^{-1} x)dx$ is equal to

a) $\frac{1}{1+x^2}+c$ b) $\tan^{-1}x+c$ c) $x\tan^{-1} x-\frac{1}{2}\log\left|1+x^2\right|+c$ d) none of these

viii) $\underset{n\to\infty}{Lim}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)$ is equal to

a) $-\frac{1}{2}$ b) 0 c) $\frac{1}{2}$ d) 1

ix) If f and g are continuous functions in [0,1] satisfying $f(x)=f(a-x)$ and $g(x)+g(a-x)=a$, then $\int_0^a f(x).g(x)dx$ is equal to

a) $\frac{a}{2}$ b) $\frac{a}{2}\int_0^a f(x)dx$ c) $\int_0^a f(x)dx$ d) $a\int_0^a f(x)dx$

x) $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ is equal to

a) 1 b) 2 c) 3 d) 4

xi) $\int_1^3 |x-2|\,dx$ is equal to

a) 0 b) 1 c) 2 d) 3

xii) If $\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = 2\int_a^{\sqrt{3}} \frac{dx}{1+x^2}$, then the value of a is

a) $\frac{1}{2}$ b) $\sqrt{3}$ c) $\frac{1}{\sqrt{3}}$ d) $\frac{\pi}{3}$

xiii) Value of $\int_{-1}^{1} (x+\sqrt{x^2+1})\, dx$

a) 0 b) $\log\frac{1}{2}$ c) $\log 2$ d) $\frac{1}{2}\log 2$

xiv) $\int_0^1 \frac{d}{dx}\left(\sin^{-1}\frac{2x}{1+x^2}\right)dx$ is equal to

a) 0 b) π c) $\frac{\pi}{2}$ d) $\frac{\pi}{4}$

xv) If $\int_n^{n+1} f(x)dx = n$, then value of $\int_2^5 f(x)dx$.

a) 12 b) 10 c) 8 d) 9

xvi) If $\int_0^a \frac{1}{1+4x^2}dx = \frac{\pi}{8}$, then the value of a is

a) $\frac{\pi}{4}$ b) $\frac{1}{2}$ c) $\frac{\pi}{2}$ d) 1

xvii) If $\int f(x)dx = g(x)\ and\ \int f(x)dx = h(x)$, then

a) $h(x) + g(x) =$ constant b) $g(x) - h(x) =$ constant

c) $h(x).g(x) =$ constant d) $g(x) = h(x)$

Section-B

2] **Short answer type questions :** [Each question caries 3 marks]

i) Evaluate the following integrals

a) $\int \frac{x^6+1}{x^2+1}dx$

b) $\int \frac{1}{a^x b^x}dx$

c) $\int \frac{\tan x}{\sec x+\tan x}dx$

d) $\int \frac{xdx}{\sqrt{x}+1}$

e) $\int \frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}dx$

f) $\int \tan x \tan 2x \tan 3x dx$

g) $\int \frac{\sin(x-a)}{\sin(x-b)}dx$

h) $\int 2^{2^{2^x}} 2^{2^x} 2^x dx$

i) $\int x^x(1+\log x)dx$

j) $\int \frac{dx}{x\sqrt{x^4-1}}$

k) $\int \frac{x^2-1}{x^2+4}dx$

l) $\int \frac{x^2dx}{x^6+a^6}$

m) $\int \sqrt{\frac{x}{a^3-x^3}}dx$

n) $\int \frac{e^x dx}{\sqrt{16-e^{2x}}}$

o) $\int e^x\left(\frac{2-\sin 2x}{1-\cos 2x}\right)dx$

p) $\int_0^1 x(1-x^5)dx$

q) $\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

r) $\int_{-1}^1 e^{|x|}dx$

s) $\int_0^{1.5}[x^2]dx$

t) $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

ii) If $f'(x) = x^2+\sin x$ and $f(0) = 0$, find $f(x)$.

iii) If $\int g(x)dx = f(x)$, then find $\int f(x).g(x)dx$

iv) If $f(x) = x + \phi(x)$, where $\phi(x)$ is an even function, then find the value of $\int_{-1}^{1} xf(x)dx$

v) If $f(x) = f(a+x)$, then show that $\int_0^{a+t} f(x)dx$ is independent of a.

Section-C

Long answer type questions :

3] **Evaluate the following integrals :** (4 or 6 marks each)

i) $\int \frac{dx}{\tan x+\cot x+\sec x+\operatorname{cosec} x}$

ii) $\int \frac{dx}{\sin x+\sec x}$

iii) $\int \frac{dx}{x\left\{6(\log x)^2+7\log x+2\right\}}$ iv) $\int \sqrt{\sec x-1}\,dx$

v) $\int \sqrt{\frac{a-x}{a+x}}dx$ vi) $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

vii) $\int \frac{dx}{3+2\sin x+\cos x}$ viii) $\int \cot^{-1}\left(1-x+x^2\right)dx$

ix) $\int e^{2x}\left(\frac{\sin 4x-2}{1-\cos 4x}\right)dx$ x) $\int \frac{x^2}{(x^2+1)(x^2+4)}dx$

xi) $\int \frac{dx}{\cos x(5-4\sin x)}$ xii) $\int \frac{xdx}{x^3-1}$

xiii) $\int_0^{\pi/2} \frac{dx}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}$ xiv) $\int_0^{\pi} |\sin x+\cos x|\,dx$

xv) $\int_{-\pi/4}^{\pi/4} \log(\sin x+\cos x)dx$ xvi) $\int_0^{\pi} \frac{x\tan x}{\sec x \operatorname{cosec} x}dx$

4) If $r=2(1-\cos\theta)$, then show that $\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=8$

5) Show that $\int_0^{\pi}\frac{xdx}{1+\cos\alpha\sin x}=\frac{\pi\alpha}{\sin\alpha}$ $(0<\alpha<\pi)$

6) Prove that $\int_1^4 f(x)dx=\frac{19}{2}$, where $f(x)=|x-1|+|x-2|+|x-3|$

7) Show that $\int_0^1 \tan^{-1}\left(1-x+x^2\right)dx=\log 2$

8) Show that $\int_0^1 \frac{\log(1+x)}{1+x^2}dx=\frac{\pi}{8}\log 2$

9) Evaluate the following integrals as limit of sums :

i) $\int_0^1 \left(3x^2-5x\right)dx$ ii) $\int_0^1 \sqrt{x}dx$ iii) $\int_0^1 3^x dx$ iv) $\int_0^3 \left(2x^2+3x-5\right)dx$

v) $\int_a^b e^{mx+c}dx$

ANSWERS

Section-A

1)	i) c	ii) b	iii) a	iv) b	v) c	vi) d
	vii) a	viii) c	ix) b	x) a	xi) b	xii) c
	xiii) a	xiv) c	xv) d	xvi) b	xvii) b	

Section-B

2) i) a) $\frac{x^5}{5}-\frac{x^3}{3}+x+c$ b) $\frac{-a^{-x}b^{-x}}{-\log(ab)}+c$ c) sec$x$–tan$x$+x+c

d) $\frac{2}{3}x^{3/2}-x+2\sqrt{x}-2\log\left|\sqrt{x}+1\right|+c$ e) $\frac{1}{a^2-b^2}\log\left|a^2\sin^2 x+b^2\cos^2 x\right|+c$

f) $-\frac{1}{3}\log|\cos 3x|+\frac{1}{2}\log|\cos 2x|+\log|\cos x|\,c$

g) $x\cos(b–a)+\sin(b–a)\log|\sin(x–b)|+c$ h) $\frac{1}{(\log 2)^3}2^{2^{2^x}}+c$ i) x^x+c

j) $\frac{1}{2}\sec^{-1}\left(x^2\right)+c$ k) $x-\frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$

l) $\frac{1}{3a^3}\tan^{-1}\left(\frac{x^3}{a^3}\right)+c$ m) $\frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)+c$

n) $\sin^{-1}\left(\frac{e^x}{4}\right)+c$ o) $C-e^x\cot x$ p) $\frac{1}{42}$

q) π r) $2(e–1)$ s) $2-\sqrt{2}$ t) $\frac{\pi}{2}$

ii) $\frac{x^3}{3}-\cos x+1$

iii) $\frac{1}{2}\{f(x)\}^2+c$

iv) $\frac{2}{3}$

Section-C

3) i) $\frac{1}{2}(\sin x-\cos x-x)+c$ ii) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}\right|+\tan^{-1}(\sin x+\cos x)+c$

iii) $\log\left|\dfrac{2\log x+1}{3\log x+2}\right|+c$

iv) $-\log\left|\left(\cos x+\dfrac{1}{2}\right)+\sqrt{\cos^2 x+\cos x}\right|+c$

v) $a\sin^{-1}\left(\dfrac{x}{a}\right)+\sqrt{a^2-x^2}+c$

vi) $x\tan^{-1}\sqrt{\dfrac{x}{a}}-\sqrt{ax}+a\tan^{-1}\sqrt{\dfrac{x}{a}}+c$

vii) $\tan^{-1}\left(1+\tan\dfrac{x}{2}\right)+c$

viii) $x\tan^{-1}x-\dfrac{1}{2}\log\left|1+x^2\right|-(1-x)\tan^{-1}(1-x)+\dfrac{1}{2}\log\left|1+(1-x)^2\right|+c$

ix) $\dfrac{1}{2}e^{2x}\cot 2x+c$

x) $-\dfrac{1}{3}\tan^{-1}x+\dfrac{2}{3}\tan^{-1}\left(\dfrac{x}{2}\right)+c$

xi) $\dfrac{1}{18}\log|1+\sin x|-\dfrac{1}{2}\log\left|1-\sin x\right|+\dfrac{4}{9}\log|5-4\sin x|+c$

xii) $\dfrac{1}{3}\log|x-1|-\dfrac{1}{6}\log\left|x^2+x+1\right|+\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+c$

xiii) $\dfrac{\pi}{4}\left(\dfrac{a^2+b^2}{a^3b^3}\right)$

xiv) $2\sqrt{2}$

xv) $-\dfrac{\pi}{4}\log 2$

xvi) $\dfrac{\pi^2}{4}$

9) i) $-\dfrac{3}{2}$ ii) $\dfrac{1}{4}$ iii) $\dfrac{2}{\log_e^3}$ iv) $\dfrac{93}{2}$ v) $\dfrac{e^c}{m}\left(e^{mb}-e^{ma}\right)$

CHAPTER- 8

APPLICATION OF INTEGRALS

Important points and Results :

- **Area of a curve between two ordinates :**
  Let $y = f(x)$ be a continuous function defined on $[a, b]$.

- If the curve $y = f(x)$ lies above x-axis on interval $[a, b]$, then the area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x=a$ and $x=b$ is given by

  $$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

- If the curve $y = f(x)$ lies below x-axis on interval $[a, b]$, then the area of the region bounded by the curve y = f(x), x-axis and the ordinates x = a and x = b is given by

  $$\text{Area} = -\int_a^b y\,dx = -\int_a^b f(x)\,dx$$

- **Area of a curve between two abscissa :**
  If the curve $x = f(y)$ lies to righ of y-axis on interval $[c, d]$, then the area of the region bounded by the curve $x = f(y)$, y-axis and the obscissa $y=c$ and $y=d$ is given by

  $$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

- If the curve $x=f(y)$ lies to left of y-axis on interval $[c, d]$, then the area of the region bounded by the curve $x=f(y)$, y-axis and the abscissa $y=c$ and $y=d$ is given by

  $$\text{Area} = -\int_c^d x\,dy = -\int_c^d f(y)\,dy$$

- **Area between two curves :**
- The area of the region enclosed between two curves $y=f(x)$, $y=g(x)$ and the lines $x=a$, $x=b$ is given by

  $$\text{Area} = \int_a^b \left[f(x) - g(x)\right]dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b]$$

  If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a<c<b$, then

  $$\text{Area} = \int_a^c \left[f(x) - g(x)\right]dx + \int_c^b \left[g(x) - f(x)\right]dx$$

Exercise - 8

Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) Multiple choice type questions :

i) The area enclosed by the circle $(x-2)^2+y^2=4$ is

a) 2π sq. units b) 4π sq. units c) 8π sq. units d) $4\pi^2$ sq. units

ii) The area bounded by the curve $y=4x-x^2$ and the x-axis is

a) $\frac{30}{7}$ sq. units b) $\frac{31}{7}$ sq. units c) $\frac{32}{3}$ sq. units d) $\frac{34}{4}$ sq. units

iii) The area bounded by the parabola $y^2 = 8x$, the x-axis and the latusrectum is

a) $\frac{32}{3}$ sq. units b) $\frac{16\sqrt{2}}{3}$ sq. units c) $\frac{16}{3}$ sq. units d) $\frac{23}{3}$ sq. units

iv) The area enclosed by the line $y=-x$, y -axis and the line $x = 3$ is

a) 6 sq. units b) 3 sq. units c) 9 sq. units d) $\frac{9}{2}$ sq. units

v) The area bounded by the parabola $y^2=4ax$ and the ordinate $x = a$ is

a) $\frac{4a^2}{3}$ sq. units b) $\frac{4a}{3}$ sq. units c) $\frac{8a^2}{3}$ sq. units d) none of these.

vi) The area enclosed by $y = a\sin x$, x-axis, $x = 0$ and $x = \pi$

a) a sq. unit b) $2a$ sq. units c) $3a$ sq. units d) none of these.

vii) The area enclosed by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

a) 12π sq. units b) 6π sq.units c) 3π sq. units d) π sq. units.

viii) The area of the region bounded by the curve $y = e^x$, $x = 1$, $x = e$ and x-axis (in sq. units) is

a) $e-1$ b) $1-e$ c) e^e-e d) $1-e^e$

ix) The area of the region bounded by the curve $y=\log_e^x$, x-axis, $x=1$ and $x = e$ (in sq. units) is

a) $2e$ b) 1 c) $2e-1$ d) $2e+1$

x) If the area enclosed by the curve $x^2=y$, x–axis, $x=1$ and $x = k$ $(k>1)$ is $\frac{26}{3}$ sq. units, then the value of k is

a) 2 b) 3 c) 4 d) none of these.

Section-B

2] Short answer type questions : [Each question caries 3 marks]

i) Find the area of the region bounded by the curve $y = |x|$, $x = -1$, $x = 1$ and the x-axis.

ii) Find the area of the region bounded by $x = \sqrt{y}$, y-axis and $y = 1$

iii) Find the area of the region enclosed by $y = \tan x$, x-axis and $x = \frac{\pi}{4}$.

iv) Find the area of the region enclosed by the curve $xy = 1$, x-axis and the ordinates $x = 1$ and $x = e$.

v) Using integration, find the area of the region bounded by the line $y - 1 = x$, the x-axis and the ordinates $x = -2$ and $x = 3$.

B] Long answer type questions : (4 or 6 marks)

3) Using integration, find the area of the region bounded by the lines $x - 2y + 4 = 0$, $x = 3$, $x = 6$ and above the x-axis.

4) Find the area of the triangle, whose sides are $y = 4x+5$, $x+y = 5$ and $4y = x+5$ respectively.

5) Using integration, find the area of ΔABC whose vertices are A(2, 1), B(3, 4) and C(5, 2).

6) Find the area of the region $\{(x, y) : y^2 \le 6ax\}$ and $\{(x, y) : x^2 + y^2 \le 6a^2\}$.

7) Show that, the area of the region enclosed by the curves $y = |x| - 1$ and $y = -|x| + 1$ is 2 sq. units.

8) Show by integration, the area of the region enclosed by the curves $x = 0$, $y = 0$ and $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $\frac{a^2}{6}$ sq. units.

9) If the area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a>0)$ is 1 square unit, then find the value of a.

10) The slope of the tangent of the curve $y = f(x)$ at the point $(x, f(x))$ is $2x+1$. If the curve passes through the point (1, 2), then find the area of the region enclosed by the curve, x-axis and the ordinate $x = 1$.

11) Using integration, find the area of the region bounded by the curves $y^2 = 2x+1$ and $x - y - 1 = 0$.

12) Draw a rough sketch and find the area of the region $\{(x, y) : x^2 + y^2 \le 2ax,\ y^2 \ge ax,\ x \ge 0,\ y \ge 0\}$.

13) Find the area of the region $\{(x, y) : x^2 + y^2 \le 4,\ x + y \ge 2\}$.

14) Find the area of the region bounded by the curve $y = x^3$, the lines $y = x+6$ and $y = 0$.

15) Find the area enclosed by the curve $x = 3\cos t$ and $y = 2\sin t$.

ANSWERS

Section-A

1) i) b ii) c iii) c iv) d v) c vi) b
vii) a viii) c ix) b x) b

Section-B

2) i) 1 sq. unit ii) $\frac{2}{3}$ sq. units iii) $\frac{1}{2}\log 2$ sq. units iv) 1 sq. units v) $\frac{17}{2}$ sq. units

Section-C

3) $\frac{51}{4}$ sq. units

4) $\frac{15}{2}$ sq. units

5) 4 sq. units

6) $\frac{4a^2}{3}\left(4\pi+\sqrt{3}\right)$ sq. units

9) $a=\frac{1}{\sqrt{3}}$

10) $\frac{5}{6}$ sq. units

11) $\frac{16}{3}$ sq. units

12) $\left(\frac{\pi a^2}{4}-\frac{2}{3}a^2\right)$ sq. units

13) $(\pi-2)$ sq. units

14) 28 sq. units

15) 6π sq. units.

CHAPTER- 9

DIFFERENTIAL EQUATION

Important points and Results :

- **Definiton :**
  An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.

- Order of a differential equation is the order of the highest order derivative occurring in the differential equation.
- Degree of a differential equaton is defined if it is a polynomial equation in its derivatives.

- Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
- The followings are some examples of defferential equations :

  i) $x\dfrac{dy}{dx}=2y$ and $xdy+y^2dx=dx$ are of first order and first degree.

  ii) $\dfrac{d^2y}{dx^2}=\dfrac{dy}{dx}$ is of second order and first degree.

  iii) $y^2\left(\dfrac{dy}{dx}\right)^2-x\dfrac{dy}{dx}=x^2$ is of first order and second degree.

  iv) $\left(\dfrac{d^2y}{dx^2}\right)^2=x\left(\dfrac{dy}{dx}\right)^3$ is of second order and second degree

  v) $x^2\dfrac{d^3y}{dx^3}-2y\dfrac{dy}{dx}=x$ is of third order and first degree.

- A functon which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equaton is called a general solution and the solution free from arbitrary constants is called particular solution.

  To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and the eliminate the arbitrary constants.

- **Differential equations of the first order and first degree :**
  A differential equation of the first order and first degree is represented in the form $Mdx+Ndy=0$ where both M and N are functions of x and y or constants. The general solution of such type of equation involves only one independent arbitrary constant.

Variable separable method is used to solve such an equation in which variables can be separated completely i.e. terms containing y should remain with dy and terms constaining x should remain with dx.

- A differential equation which can be expressed in the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$ where $f(x,y)$ and $g(x,y)$ are homogeneous functions of degree zero is called a homogeneous differential equation.

- A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only is called a first order linear differential equation.

- **Methods of solving first order, first degree differential equation :**
  In this section we shall discuss three methods of solving first order first degree differential equations.

- **Differential equations with variables separable :**
  A first order - first degree differential equation is of the form

  $\frac{dy}{dx} = F(x, y)$ .....(1)

  If $F(x,y)$ can be expressed as a product $g(x)$ $h(y)$, where $g(x)$ is a function of x and $h(y)$ is a function of y, then the differential equation (1) is said to be of variable separable type.
  The differential equation (1) then has the form

  $\frac{dy}{dx} = h(y).g(x)$ ....(2)

  If $h(y) \neq 0$, separating the variables (2) can be rewritten as

  $\frac{1}{h(y)} dy = g(x)dx$ .... (3)

  Integrating both sides of (3), we get,

  $\int \frac{1}{h(y)} dy = \int g(x)dx$ .... (4)

  Thus (4) provides the solutions of given differential equation in the form
  $H(y) = G(x) + C$

  Here $H(y)$ and $G(x)$ are the anti derivaties of $\frac{1}{h(y)}$ and $g(x)$ respectively and C is the arbitrary constant.

- **Homogeneous function and Homogeneous differential equation :**
  A function $F(x, y)$ is called homogeneous function of degree n if $F(\lambda x, \lambda y) = \lambda^n F(x,y)$, where λ is non-zero real number.

A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is called homogeneous differential equation, if $F(x, y)$ is a homogeneous function of degree zero i.e. $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$

For an example :

$(x^2+xy)\,dy = (x^2+y^2)dx$

or $\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$

In homogeneous differential equation because here

$$F(x, y) = \frac{x^2+y^2}{x^2+xy}$$

$$F(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\lambda^2 x^2 + \lambda x.\lambda y}$$

$$= \frac{\lambda^2\left(x^2+y^2\right)}{\lambda^2\left(x^2+xy\right)}$$

$$= \lambda^0 F(x, y)$$

Hence $F(x, y)$ is homogeneous function of degree zero.

$\therefore \frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$ is a homogeneous differential equation.

a) To solve a homogeneous differential equation of the type

$$\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right) \text{..(1)}$$

We make the substitution $y=v.x$.....................................(2)

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{..(3)}$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get,

$$v + x\frac{dv}{dx} = g(v)$$

or, $x\frac{dv}{dx} = g(v) - v$ ....(4)

Separating the variables in equation (4), we get

$$\frac{dv}{g(v)-v} = \frac{dx}{x} \text{..(5)}$$

Integrating both sides of equation (5), we get

$$\int \frac{dv}{g(v)-v} = \int \frac{1}{x}\,dx + C \text{ ....(6)}$$

Equation (6) gives general solution of the differential equation (1) when we replace v by $\frac{y}{x}$.

b) If the homogeneous differential equation is in the form

$$\frac{dx}{dy} = F(x, y) \text{ ....(1)}$$

where $F(x, y)$ is homogenous function of degree zero, then we make substitution

$$\frac{x}{y} = v \text{ i.e. } x = vy \text{...(2)}$$

Differentiating equation (2) with respect to y.

We get, $\frac{dx}{dy} = v + y\frac{dv}{dy}$ and we proceed further to find the general solution as discussed above by writing $\frac{dx}{dy} = F(x, y) = h\left(\frac{x}{y}\right)$

- **Linear differential equations :**

A linear differential equation is in the form of $\frac{dy}{dx} + Py = Q$ where P, Q are constant or functions of x only.

Here, the integrating factor is $(I.F) = e^{\int Pdx}$.

Now, we can write the solution of the given differential equation as

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

or, $$y.e^{\int Pdx} = \int \left(Q.e^{\int Pdx}\right)dx + C$$

Sometimes, we get linear differential equation is in the form of $\frac{dx}{dy} + P_1 x = Q_1$, where P_1 and Q_1 are constants or functions of y only.

Then I.F = $e^{\int Pdy}$ and the solution of the differential equation is given by

$$x.(I.F) = \int (Q_1 \times I.F)\,dy + C$$

or, $$x.e^{\int Pdy} = \int \left(Q_1.e^{\int Pdy}\right)dy + C$$, C = Integration Constant

Exercise-9

Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) **Multiple choice type questions : (choose the correct option)**

i) The order of differential equation $\left(\frac{d^4y}{dx^4}\right)^3+\frac{d^3y}{dx^3}=\sqrt{1+\frac{dy}{dx}}$ is

a) 6 b) 4 c) 3 d) 7

ii) The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2+\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^4+\frac{dy}{dx}+y=6x^3$ is

a) 4 b) 3 c) 2 d) 1

iii) The order and degree of the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}+x^{\frac{1}{5}}=0$ respectively, are

a) 2 and 4 b) 2 and 2 c) 2 and 3 d) 3 and 3

iv) Which of the following is a second order differential equation

a) $(y')^2+x=y^2$ b) $y'y''+y=\sin x$ c) $y'''+(y'')^2+y=0$ d) $y'=y^2$

v) The differential equation $y\frac{dy}{dx}+x=C$ represents

a) family of hyperbolas b) family of parabolas

c) family of ellipses d) family of circles.

vi) The order and degree of differential equation $\left(\frac{d^3y}{dx^3}\right)^2-3\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^4=y^4$ are

a) 1, 4 b) 3, 4 c) 2, 4 d) 3, 2

vii) If $y=e^{-x}(A\cos x+B\sin x)$, then y is a solution of

a) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ b) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$

c) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$ d) $\frac{d^2y}{dx^2}+2y=0$

viii) The solution of differential equation $xdy-ydx=0$ represents

a) a rectangular hyperbola b) parabola whose vertex is at origin

c) a straight line passing through origin d) a circle whose centre is at origin.

ix) The number of solutions of $\frac{dy}{dx}=\frac{y+1}{x-1}$, when $y(1)=2$ is

a) one b) two c) infinite d) none of these

x) The solution of differential equation $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$ is

a) $y=\tan^{-1}x$ b) $y-x=K(1+xy)$ c) $x=\tan^{-1}y$ d) $\tan(xy)=K$

xi) The integrating factor of differential equation $\frac{dy}{dx}+y\tan x-\sec x=0$ is

a) $\cos x$ b) $\sec x$ c) $e^{\cos x}$ d) $e^{\sec x}$

xii) The integrating factor of $x\frac{dy}{dx}-y=x^4-3x$ is

a) x b) $\log x$ c) $\frac{1}{x}$ d) $-x$

xiii) The solution of $\frac{dy}{dx}+y=e^{-x}$, $y(0)=0$ is

a) $y=e^x(x-1)$ b) $y=xe^{-x}$ c) $y=xe^{-x}+1$ d) $y=(x+1)e^{-x}$

xiv) The general solution of $e^x\cos y\,dx-e^x\sin y\,dy=0$ is

a) $e^x\cos y=K$ b) $e^x\sin y=K$ c) $e^x=K\cos y$ d) $e^x=K\sin y$

xv) The general solution of $\frac{dy}{dx}=2xe^{x^2-y}$ is

a) $e^{x^2-y}=C$ b) $e^{-y}+e^{x^2}=C$ c) $e^y=e^{x^2}+C$ d) $e^{x^2+y}=C$

2] Very short answer type questions : (1 or 2 marks each)

i) Find the order and degree of differential equation :

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]=\frac{d^2y}{dx^2}$$

ii) Solve : $e^{\frac{dy}{dx}}=x^2$

iii) Determine the order and degree of the following differential equation :

$$y=x\frac{dy}{dx}+\sqrt{a^2\left(\frac{dy}{dx}\right)^2+b^2}$$

iv) Find the differential equaton of $y=Ax+\frac{B}{x}$, where A and B are arbitrary constants.

v) Solve : $\frac{dy}{dx}=1-x+y-xy$

vi) Find the integrating factor of the differential equations :

$$\left(1+x^2\right)\frac{dy}{dx}+y=\tan^{-1}x$$

vii) Solve : $\frac{dy}{dx}-\frac{y(x+1)}{x}=0$

viii) Solve : $\frac{dy}{dx}=e^{x-y}+1$

ix) Find the integrating factor of the differential equation $\frac{dy}{dx}+y=\frac{1+y}{x}$

x) Solve : $(\cos y+y\cos x)dx+(\sin x-x\sin y)dy=0$

Section - B

3] Short answer type questions : (3 marks each)

1) Show that $v=\frac{A}{r}+B$ satisfyies the differential equation : $\frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=0$

2) From the differential equation corresponding to $y^2-2ay+x^2=a^2$ by eliminating a.

3) Solve : $x\sqrt{1-y^2}\,dx+y\sqrt{1-x^2}\,dy=0$

4) Solve : $\frac{dy}{dx}=\frac{2x-3y}{3x-2y}$

5) Find the solution of differential equation $(2y-1)dx-(2x+3)dy=0$

6) Find the general solution of $\frac{dy}{dx}+y\tan x=\sec x$.

7) Find the solution of differential equation $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$.

8) Solve : $x\frac{dy}{dx}+y=e^x$

9) Solve : $\frac{dy}{dx}+\frac{y}{x}=\sin x$

10) Find the integrating factor of the differential equation $\left(x\log x\right)\frac{dy}{dx}+y=2\log x$

Section - C

4] **Long answer type questions :** (For each question carries 4 or 6 marks)

1) Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right)=3x+4y$, given that y=0 when x=0.

2) Solve : $2x^2\frac{dy}{dx}-2xy+y^2=0$

3) Find the particular solution of the following differential equation :

$\frac{dy}{dx}=1+x^2+y^2+x^2y^2$, given that y=1, when x=0.

4) Find the particular solution of the differential equation :

$x\left(x^2-1\right)\frac{dy}{dx}=1$; y=0 when x=2.

5) Solve : $x\frac{dy}{dx}=y-x\tan\left(\frac{y}{x}\right)$

6) Solve : $\left(1+x^2\right)\frac{dy}{dx}+y=\tan^{-1}x$

7) Show that the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right)=0$ is homogeneous. Find the particular solution of this differential equation, given that x=1, when $y=\frac{\pi}{2}$.

8) Solve : $\frac{dy}{dx}+\frac{1}{\log x}.y=\frac{2}{x}$

9) Solve : $xy\frac{dy}{dx}-y^2=(x+y)^2\ e^{\frac{-y}{x}}$

10) Find the particular solution of the differential equation $\left(1+x^3\right)\frac{dy}{dx}+6x^2y=\left(1+x^2\right)$, given that y =1 when x =1.

11) Solve : $\frac{dy}{dx}=\frac{x-y+1}{2x-2y+3}$

12) Solve : $y-x\frac{dy}{dx}=2(1+x^2\frac{dy}{dx})$ Given y =1, when x=1.

13) Show that the differential equation $\left(xe^{\frac{y}{x}}+y\right)dx=xdy$ is homogeneous. Find the particular solution of this differential equation, given that x =1, when y =1.

14) Find the particular solution of the differential equation $\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y,\ (y \neq 0)$ given that $x = 0$, when $y = \frac{\pi}{2}$.

15) Find the equation of a curve passing through origin and satisfying the differential equation $\left(1+x^2\right)\frac{dy}{dx} + 2xy = 4x^2$.

16) Solve : $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$

17) Find the general solution of $(1+\tan y)(dx-dy)+2x\,dy=0$.

18) Solve : $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$

19) Find the equation of a curve passing through (2,1), if the slope of the tangent to the curve at any point (x, y) is $\frac{x^2+y^2}{2xy}$.

20) Find the equation of the curve through the point (1,0), if the slope of the tangent to the curve at any point (x, y) is $\frac{y-1}{x^2+x}$.

21) Solve : $x\frac{dy}{dx} = y\left(\log y - \log x + 1\right)$

22) Find the equation of a curve passing through the point (1, 1), if the tangent drawn at any point *P(x,y)* on the curve meets the coordinate axes at *A* and *B* such that *P* is the mid point of *AB*.

23) Find the general solution of $\frac{dy}{dx} - 3y = \sin 2x$.

24) Solve the differential equation $dy=\cos x(2-y\operatorname{cosec} x)dx$ given that $y=2$ when $x = \frac{\pi}{2}$.

25) Find the general solution of the differential equation $\left(1+y^2\right)+\left(x-e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$

ANSWERS

Section-A

1) i) b ii) c iii) a iv) b v) d vi) d
vii) c viii) c ix) a x) b xi) b xii) c
xiii) b xiv) a xv) c

2) i) order = 2 and degree = 1 ii) $y = 2(x\log x - x) + c$ iii) order = 1 and degree = 2

iv) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$ v) $\log|1+y| = x - \dfrac{x^2}{2} + c$ vi) $e^{\tan^{-1}x}$

vii) $y = xe^{x+c}$ viii) $e^{y-x} = x + c$ ix) $e^x \cdot \dfrac{1}{x}$

x) $x\cos y + y\sin x = c$

Section-B

3) 2) $2y^2y_1^2 + 4xyy_1 + x^2(1 - y_1^2) = 0$

3) $\sqrt{1-x^2} + \sqrt{1-y^2} = C$

4) $y^2 - 3xy + x^2 = C$

5) $\dfrac{2x+3}{2y-1} + K$, where $K = c^2$

6) $y\sec x = \tan x + c$

7) $e^y - e^x = \dfrac{x^3}{3} + C$

8) $y = \dfrac{e^x}{x} + \dfrac{K}{x}$

9) $x(y + \cos x) = \sin x + c$

10) $\log x$

Section - C

4). 1) $4e^{3x} + 3e^{-4y} = 7$

2) $\log|x| + c = \dfrac{2x}{y}$

3) $\tan^{-1} y = x + \dfrac{x^3}{3} + \dfrac{\pi}{4}$

4) $y = \dfrac{1}{2}\log\left|\dfrac{x^2-1}{x^2}\right| - \dfrac{1}{2}\log\dfrac{3}{4}$

5) $x\sin\frac{y}{x}=C$

6) $y=\left(\tan^{-1}x-1\right)+Ce^{-\tan^{-1}x}$

7) $\cos\frac{y}{x}=\log|x|$

8) $y\log x=(\log x)^2+C$

9) $(x+y)\log|xc|=xe^{\frac{y}{x}}$

10) $y\left(1+x^3\right)^2=\frac{x^6}{6}+\frac{x^4}{4}+\frac{x^3}{3}+x+\frac{9}{4}$

11) $x-2y=\log|x-y+2|+c$

12) $(y-2)^2(2x+1)^2=9x^2$

13) $e^{\frac{y}{x}}\log x-e^{\frac{y}{x}-1}+1=0$

14) $x\sin y=y^2\sin y-\frac{\pi^2}{4}$

15) $y=\frac{4x^3}{3(1+x^2)}$

16) $y=-\cos x+\frac{2\sin x}{x}+\frac{2\cos x}{x^2}+\frac{x}{3}\log x-\frac{x}{9}+Cx^{-2}$

17) $x(\sin y+\cos y)=\sin y+Ce^{-y}$

18) $\log\left|1+\tan\frac{x+y}{2}\right|=x+c$

19) $2(x^2-y^2)=3x$

20) $(y-1)(x+1)+2x=0$

21) $\log\left(\frac{y}{x}\right)=Cx$

22) $xy=1$

23) $y=-\frac{1}{13}(2\cos 2x+3\sin 2x)+Ce^{3x}$

24) $y\sin x=-\frac{1}{2}\cos 2x+\frac{3}{2}$

25) $2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+K$

CHAPTER- 10

VECTOR ALGEBRA

Some Important points and Results :

- **Vector :**

  A quantity that has magnitude as well as direction is called a vector.

Directed line segment is a vector, denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$ and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

The point A from where the vector $\overrightarrow{AB}$ starts is called its initial point, and the point B where it ends is called its terminal point. The distance between initial and terminal point of a vector is called the magnitude (or length) of the vector, denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$.

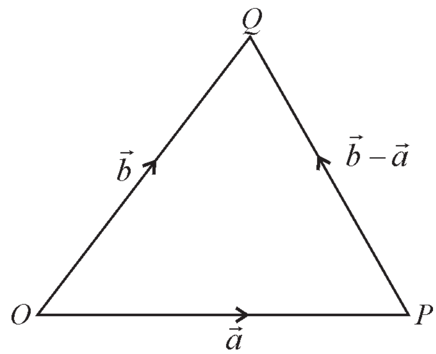
- **Position Vector :**

  Let O be an arbitary point. Then the position vector of the point P w.r.t O is the vector $\overrightarrow{OP}$. The initial point O is called the origin of reference.

  Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points P and Q respectively referred to the origin O.

  $\therefore$ PQ = (Position vector Q) – (Position vector of P)

  $= \vec{b} - \vec{a}$

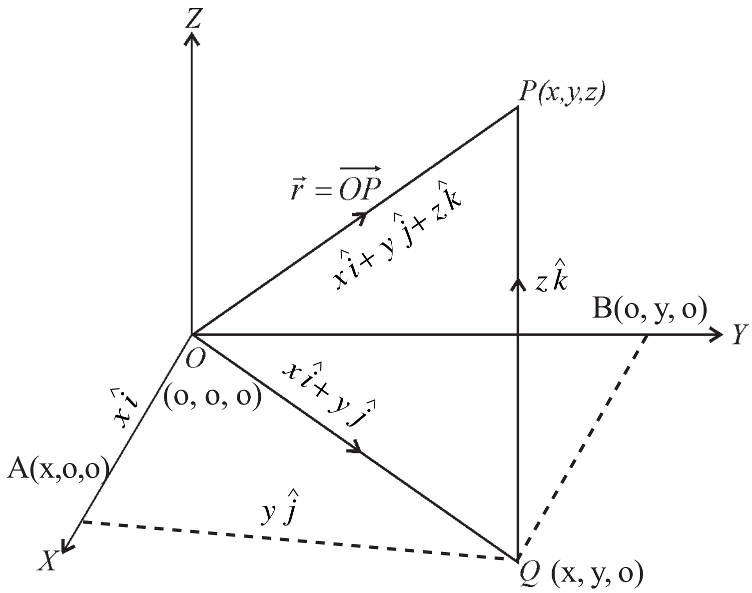


If $P(x, y, z)$ is a point in three dimensional coordinate space, then position vector of P w.r.t. O as origin is $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

Modulus of $\overrightarrow{OP}$

$= |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

Here, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vector along $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ respectively.



- **Section formula**

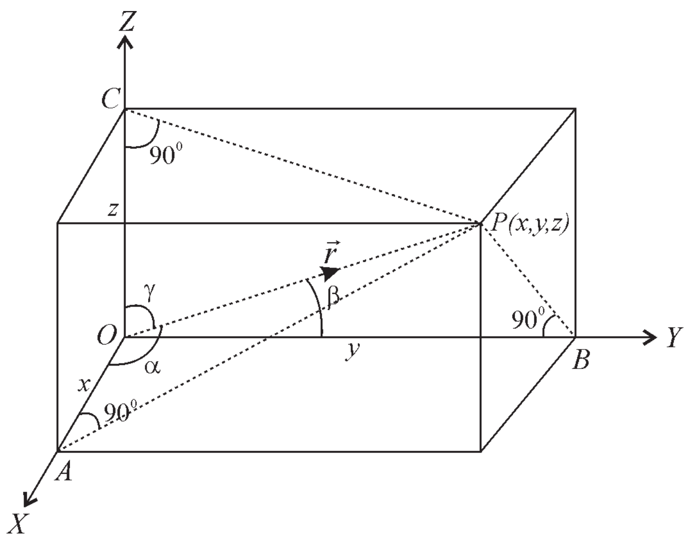
  The position vector of a point *R* dividing a line segment joining the points *P* and *Q* whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio $m:n$.

  i) internally, is given by $\dfrac{m\vec{b}+n\vec{a}}{m+n}$

  ii) externally, is given by $\dfrac{m\vec{b}-n\vec{a}}{m-n}$

- **Direction Cosines of Vectors :**

  Consider the position vector $\overrightarrow{OP}$ (or $\vec{r}$) of a point *P(x,y,z)* as in figure. The angles α, β, γ made by the vector $\vec{r}$ with the positive directions of *x, y* and *z*-axes respectively, are called it's **direction angles**. The cosine values of these angles i.e. cosα, cosβ and cosγ are called direction cosines of the vector $\vec{r}$ and usually denoted by *l, m* and *n*, respectively.



  In figure, the triangle *OAP* is right angled, triangle and in it, we have $\cos\alpha = \dfrac{x}{r}$. Similarly, from the right angled triangles *OBP* and *OCP*, we may write $\cos\beta = \dfrac{y}{r}$ and $\cos\gamma = \dfrac{z}{r}$.

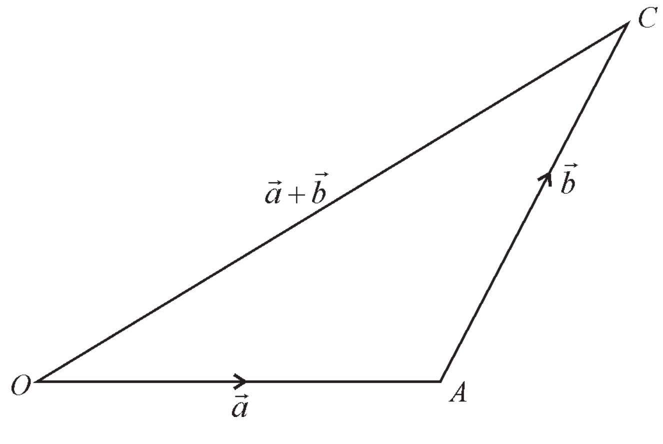
  Thus, the coordinates of the point *P* may also be expressed as (*lr, mr, nr*). The numbers *lr, mr* and *nr* are proportional to the direction cosines are called as direction ratios of vector $\vec{r}$ and denoted as *a, b* and *c* respectively.

- **Types of Vectors :**

- **Zero Vector :** A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{0}$. Zero vector can not be assinged a definite direction as it has zero magnitude or, alternatively otherwise, it may be regarded as having any direction. The vector $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vector.

- **Unit Vector :** A vector whose magnitude is unity (i.e. 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\vec{a}$ and $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.
- **Co-initial Vectors :** Two or more vectors having the same initial point are called co-initial vectors.
- **Collinear Vectors :** Two or more vectors are said to be collinear if they are parallel to the same lines, irrespective of their magnitudes and directions.
- **Equal Vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.
- **Negative of a Vector :** A vector whose magnitude is the same as that of a given vector (say $\overrightarrow{AB}$), but direction is opposite to that of it, is called negative of the given vector. Vector $\overrightarrow{BA}$ is negatives of the vector $\overrightarrow{AB}$ and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.
- **Coplanar Vectors :** Three or more vectors are said to be coplanar, when they are parallel to the same plane or lie in the same plane. If m and n are two non zero scalars, then all linear combinations $m\vec{a} + n\vec{b}$ of two given non-collinear vectors $\vec{a}$ and $\vec{b}$ are coplanar.
- **Free Vectors :** A vector drawn parallel to a given vector through a specified point in space is called a localized vector; but a vector, which depends only on its magnitude and direction and is independent of its position in space, is called a free vector.
- **Addition and Substraction of Vectors :**

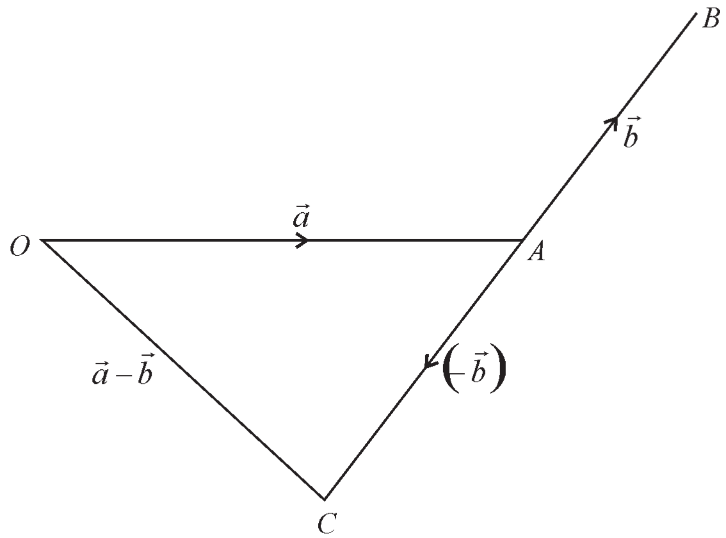
  If three points *O, A* and *C* are so chosen that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AC} = \vec{b}$; then the vector $\overrightarrow{OC}$ is called the sum of the vectors $\vec{a}$ and $\vec{b}$ i.e. $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \vec{b}$



The operation of constructing the sum of two vectors is called triangle law of addition.

The substraction of $\vec{b}$ from $\vec{a}$ is defined as the addition $-\vec{b}$ to $\vec{a}$.

Thus $\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right)$

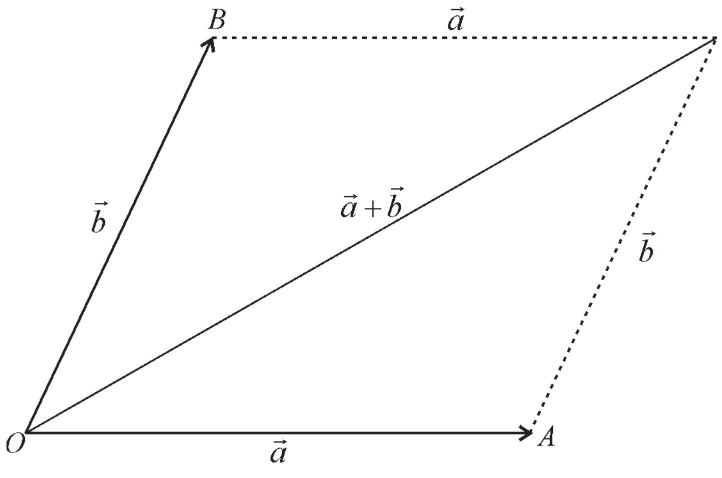


We have, $\vec{a}-\vec{b}=\overrightarrow{OA}-\overrightarrow{AB}=\overrightarrow{OA}+\left(-\overrightarrow{AB}\right)$

$$=\overrightarrow{OA}+\overrightarrow{AC} \quad \left[\because \overrightarrow{AB}=-\overrightarrow{AC}\right]$$

$$=\overrightarrow{OC}$$

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of prallelogram in magnitude and direction (in fig.), then their sum $\vec{a}+\vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the **parallelogram law of vector addition**.



- **Properties of Vector addition :**

i) For any two vectors $\vec{a}$ and $\vec{b}$,

$\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative property)

ii) For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$ (Associative property)

For any vector $\vec{a}$, we have $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$.

Here, the zero vector $\vec{0}$ is called the additive identity for the vector addition.

- **Multiplication of a vector by a scalar :**

1) If $\vec{a}$ is vector and λ is a scalar, then their product denoted by $\lambda\vec{a}$ or $\vec{a}\lambda$, is the vector

   i) Which is collinear with $\vec{a}$.

   ii) Whose magnitude is equal to $|\vec{a}||\lambda|$, where $|\vec{a}|$ and $|\lambda|$ are the moduli of $\vec{a}$ and λ -respectively.

   iii) Whose direction is the same as that of $\vec{a}$, when λ>0 and opposite to $\vec{a}$ and when λ<0.

   If λ=0 or $\vec{a}=\vec{0}$, then the product $\lambda\vec{a}$ is a zero vector. Again if $\hat{a}$ is the unit vector in the direction of $\vec{a}$, then $\vec{a}=|\vec{a}|\hat{a}$ or $\hat{a}=\dfrac{\vec{a}}{|\vec{a}|}$.

2] If m and n are scalars and $\vec{a}$ is a vector then from the definition of multiplication of a vector by a scalar it follows that $m(n\vec{a})=(mn)\vec{a}=n(m\vec{a})$ and $(m+n)\vec{a}=m\vec{a}+n\vec{a}$.

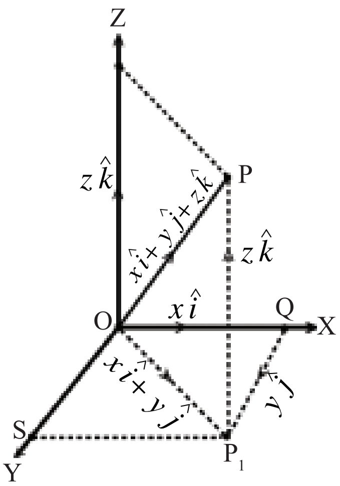
- **Components of a vector :**

Consider the position vector $\overrightarrow{OP}$ of a point *P(x,y,z)* as in Fig; let P_1 be the foot of the perpendicular from P on the plane XOY, we thus, see that P_1P is parallel to Z-axis. As $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the *x, y* and *z* axes respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P}=\overrightarrow{OR}=z\hat{k}$. Similarly, $\overrightarrow{QP_1}=\overrightarrow{OS}=y\hat{j}$ and $\overrightarrow{OQ}=x\hat{i}$.

Therefore, it follows that

$$\overrightarrow{OP_1}=\overrightarrow{OQ}+\overrightarrow{QP_1}=x\hat{i}+y\hat{j}$$

and $$\overrightarrow{OP}=\overrightarrow{OP_1}+\overrightarrow{P_1P}=x\hat{i}+y\hat{j}+z\hat{k}$$



Hence, the position vector of P with reference to O is given by $\overrightarrow{OP}(or\ \vec{r})==x\hat{i}+y\hat{j}+z\hat{k}$.

This form of any vector is called its component form. Here *x, y* and *z* are called the scalar components of $\vec{r}$ and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the vector components of $\vec{r}$ along the respective axes.

- The length of any vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ is readily determined by applying the Pythagorous Theorem twice. We note that in the right angle triangle OQP_1

$$\left|\overrightarrow{OP_1}\right|=\sqrt{\left|\overrightarrow{OQ}\right|^2+\left|\overrightarrow{QP_1}\right|^2}=\sqrt{x^2+y^2}$$

From triangle OPP_1, $\left|\overrightarrow{\mathrm{OP}}\right|=\sqrt{(\mathrm{P}\vec{\mathrm{P}}_1)2+\mathrm{b}\vec{\mathrm{P}})^2}=\sqrt{z^2+x^2+y^2}$

∴ The length of any vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ is given by $|\vec{r}|=\left|x\hat{i}+y\hat{j}+z\hat{k}\right|=\sqrt{x^2+y^2+z^2}$.

- If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ respectively; then

  i) The sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

  $$\vec{a}+\vec{b}=(a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$$

  ii) The difference of the voclor $\vec{a}$ and $\vec{b}$ is given by $\vec{a}-\vec{b}=(a_1-b_1)\hat{i}+(a_2-b_2)\hat{j}+(a_3-b_3)\hat{k}$.

  iii) The vectors $\vec{a}$ and $\vec{b}$ are equal if and only if $a_1=b_1$, $a_2=b_2$ and $a_3=b_3$

  iv) The multiplication of vector $\vec{a}$ by any scalar λ is given by $\lambda\vec{a}=(\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}$.

  v) $\vec{a}$ and $\vec{b}$ vectors are collinear if and only if $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=\lambda\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$

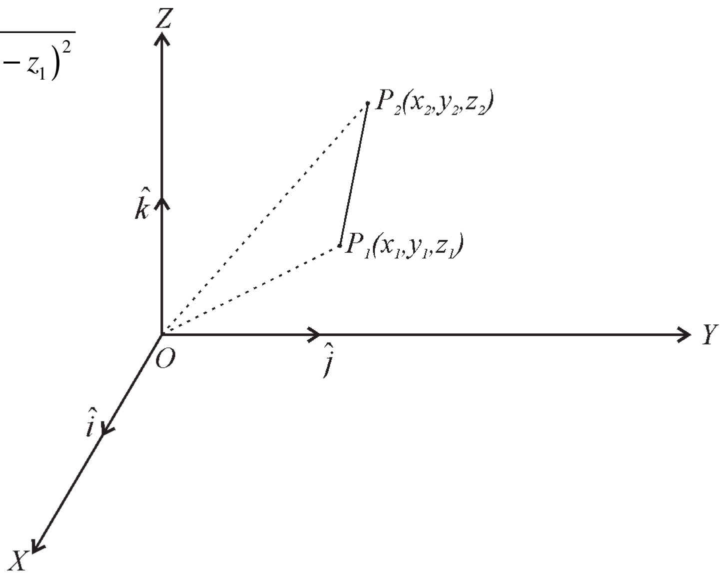
  $$\Leftrightarrow b_1=\lambda a_1,\ b_2=\lambda a_2,\ b_3=\lambda a_3$$

  $$\Leftrightarrow \frac{b_1}{a_1}=\frac{b_2}{a_2}=\frac{b_3}{a_3}=\lambda$$

  vi) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; then a_1, a_2, a_3 are called direction ratios of $\vec{a}$.

- **Vector joining two points :**
  If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points; then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$. The magnitude of vector $\overrightarrow{P_1P_2}$ is given by

  $$\left|\overrightarrow{P_1P_2}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$



- **Scalar (or dot) product of two vectors :**
  The scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$ is defined as $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$. Where θ is the angle between $\vec{a}$ and $\vec{b}$, $0\le\theta\le\pi$.

  i) Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then $\vec{a}.\vec{b}=0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other i.e. $\vec{a}.\vec{b}=0\Leftrightarrow\vec{a}\perp\vec{b}$.

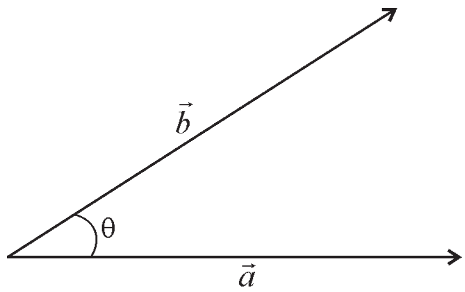
  ii) If $\theta=0$ then $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|$.

iii) If $\theta = \pi$; then $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

iv) for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$; we have

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$$

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$$



v) The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \text{ or } \theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

vi) The scalar product is commulative i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

(vii) If two vectors $\vec{a}$ and $\vec{b}$ are given in component form as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then scalar product is given by $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

- **Projection of a vector on a lines :**

Suppose a vector $\overrightarrow{AB}$ makes an angle θ with a given directed line *l* (say); in the anticlockwise direction. Then the projection of $\overrightarrow{AB}$ on *l* is a vector $\vec{p}$ (say) with magnidute $|\overrightarrow{AB}||\cos\theta|$ and the direction of $\vec{p}$ being the same (or opposite) to that of the line *l*, depending upon whether cosθ is positive or negative. The vector $\vec{p}$ is called the projection vector, and its magnitude $|\vec{p}|$ is simply called as the projection of the vector $\overrightarrow{AB}$ on the directed line *l*.

i) If $\hat{p}$ is the unit vector along a line *l*, then the projection of a vector $\vec{a}$ on the line *l* is given by $\vec{a}.\hat{p}$.

ii) Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by $\vec{a}.\left(\frac{\vec{b}}{|\vec{b}|}\right)$, or $\frac{1}{|\vec{b}|}(\vec{a}.\vec{b})$.

iii) If θ=0; then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if θ=π; then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.

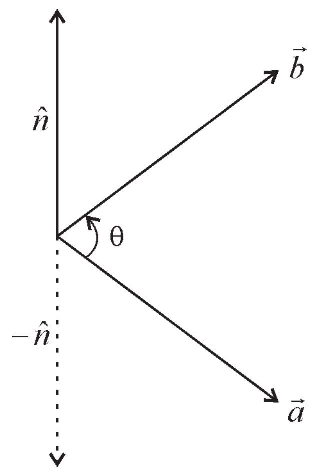
iv) If $\theta = \pi/2$ or $\theta = 3\pi/2$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

- **Vector (or cross) product of two vectors :**

The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and defined as

$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$.

Where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0\le\theta\le\pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.



i) Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then $\vec{a}\times\vec{b}=0$ if and ony if $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other i.e. $\vec{a}\times\vec{b}=\vec{0}\Leftrightarrow\vec{a}\,||\,\vec{b}$

ii) If $\theta=\dfrac{\pi}{2}$ then $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|$

iii) For mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ we have

$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$

$\hat{i}\times\hat{j}=\hat{k},\ \ \hat{j}\times\hat{k}=\hat{i},\ \ \hat{k}\times\hat{i}=\hat{j}$

iv) In terms of vector product; the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as

$$\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

v) It is always true that the vector product is not commutative, as $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$. Indeed, $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

vi) $\hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{j}=-\hat{i}$ and $\hat{i}\times\hat{k}=-\hat{j}$

vii) If $\vec{a}$ and $\vec{b}$ represent the two adjacent sides of a triangle then its area is given as $\dfrac{1}{2}|\vec{a}\times\vec{b}|$ sq. units.

viii) If $\vec{a}$ and $\vec{b}$ represent the two adjacent sides of a parallelogram, then its area is given by $|\vec{a}\times\vec{b}|$ sq. units.

ix) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors and λ be a scalar, then

i) $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

ii) $\lambda\left(\vec{a}\times\vec{b}\right)=(\lambda\vec{a})\times\vec{b}=\vec{a}\times\left(\lambda\vec{b}\right)$

x) Let $\vec{a}$ and $\vec{b}$ two vectors given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ respectively. Then their cross product may be given by

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

- The magnitude (r), direction ratios (a, b, c) and direction cosines (l, m, n) of any vector are related as :

  $l=\dfrac{a}{r},\ m=\dfrac{b}{r},\ n=\dfrac{c}{r}$

- The scalar components of a vector are its direction ratios, and represent its projection along the respective axes.
- The vector sum of the three sides of a triangle taken in order is $\vec{0}$.
- The vector sum of two co-initial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.

Exercise-10

Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) Multiple choice type questions :

i) The vector joining the points A(–2, 1, 0) and B(–1, 2, 3) directed from A to B is

a) $\hat{i}-3\hat{j}+3\hat{k}$ b) $-3\hat{i}-\hat{j}+3\hat{k}$ c) $3\hat{i}-\hat{j}-3\hat{k}$ d) $\hat{i}+\hat{j}+3\hat{k}$

ii) The direction cosines of the vector joining the points (3, –1, 2) and (5, –4, 2) are

a) $\left(\dfrac{2}{\sqrt{13}},\dfrac{-3}{\sqrt{13}},0\right)$ b) $\left(\dfrac{3}{\sqrt{13}},\dfrac{2}{\sqrt{13}},0\right)$

c) $\left(\dfrac{-2}{\sqrt{13}},\dfrac{-3}{\sqrt{13}},0\right)$ d) none of these.

iii) Unit vector in the direction of vector $\overrightarrow{AB}$ where A and B are the points (5, –3, 2) and (3, 3, 5) is

a) $-\dfrac{2}{7}\hat{i}+\dfrac{6}{7}\hat{j}+\dfrac{3}{7}\hat{k}$ b) $\dfrac{8\hat{i}+\hat{j}+7\hat{k}}{\sqrt{113}}$

c) $\frac{8\hat{i}+7\hat{k}}{\sqrt{113}}$ d) $\frac{2}{7}\hat{i}+\frac{6}{7}\hat{j}+\frac{3}{7}\hat{k}$

iv) The position vector of the point which divides the line joining the points (1, 2, –1) and (–1, 1, 1) internally in the ratio 2:3

a) $\left(-\frac{1}{5},\frac{7}{5},\frac{1}{5}\right)$ b) $\left(-\frac{1}{5},\frac{8}{5},\frac{1}{5}\right)$ c) $\left(\frac{1}{5},\frac{8}{5},\frac{-1}{5}\right)$ d) $\left(\frac{2}{5},\frac{3}{5},\frac{-1}{5}\right)$

v) The value of λ for which $2\hat{i}+\hat{j}-3\hat{k}$ and $6\hat{i}+\lambda\hat{j}-9\hat{k}$ are collinear is –

a) $\frac{1}{3}$ b) 3 c) –3 d) none of these

vi) If $|\vec{a}|=3$ and $-1\le K\le 2$ then $|K\vec{a}|$ lies in the interval

a) [0, 6] b) [–3, 6] c) [3, 6] d) [1, 2]

vii) A regular hexagon ABCDEF is inscribed in a circle with centre O, then $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}$ equals to

a) $2\overrightarrow{AO}$ b) $4\overrightarrow{AO}$ c) $6\overrightarrow{AO}$ d) $6\overrightarrow{OA}$

viii) A vector $\vec{r}$ is of magnitude $3\sqrt{2}$ units which makes angles $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with y and z axes respectively. The value of $\vec{r}$ is

a) $\hat{i}+\hat{j}$ b) $\pm\frac{1}{\sqrt{2}}\left(\hat{i}+\hat{j}\right)$ c) $\pm\left(3\hat{i}+3\hat{j}\right)$ d) $\pm 3\hat{i}+3\hat{j}$

ix) The value of $\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{i}+\hat{k}\right)+\hat{k}.\left(\hat{i}+\hat{j}\right)$ is

a) 2 b) –1 c) 3 d) 1

x) If $|\vec{a}|=2,\ |\vec{b}|=3,\ |\vec{c}|=4$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ then $\vec{b}.\vec{c}+\vec{c}.\vec{a}+\vec{a}.\vec{b}$ is equal to

a) $\frac{29}{2}$ b) $-\frac{29}{2}$ c) $\frac{19}{2}$ d) $-\frac{19}{2}$

2] **Very short answer type questions :** [1 or 2 marks for each question]

i) Show that the vectors $-\hat{i}+\hat{j}$, $-4\hat{i}-6\hat{j}$ and $5\hat{i}+5\hat{j}$ are the sides of a right-angled triangle.

ii) If $\vec{a}=4\hat{i}-3\hat{j}$ and $\vec{b}=-2\hat{i}+5\hat{j}$ are the positoin vectors of the points A and B respectively; find the position vector of the middle point of the line segment $\overrightarrow{AB}$.

iii) The position vectors of the points A and B are $3\hat{i}-\hat{j}+7\hat{k}$ and $4\hat{i}-3\hat{j}-\hat{k}$; find the magnitude and direction cosines of $\overrightarrow{AB}$.

iv) Show that the vectors $\vec{\alpha}=-3\hat{i}-2\hat{j}+\hat{k}$ and $\vec{\beta}=-2\hat{i}+\hat{j}-4\hat{k}$ are perpendicular to each other.

vi) If $\vec{a}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}-2\hat{k}$, find the angle between the vectors $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}+\vec{b}\right)$

vi) If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}.\vec{b}| = |\vec{a}\times\vec{b}|$, then find the angle between the vectors $\vec{a}$ and $\vec{b}$.

vii) If $\vec{a} = -\hat{i}+2\hat{j}+\hat{k}$, $\vec{b} = 3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{c} = 2\hat{i}+\hat{j}+3\hat{k}$ find $\left[\vec{c}\,\vec{a}\,\vec{b}\right]$.

viii) Find the direction cosines of the straight line joining the points :

a) (2, –1, 40) and (0, 1, 5)

b) (4, 3, –5) and (–2, 1, –8)

ix) Find the cosine of the angle between the lines BA and BC where A, B, C are the points (1, 2, 3), (2, 5, –1) and (–1, 1, 2) respectively.

x) If the vectors $X\,\hat{i}-4\,\hat{j}+5\hat{k}$, $\hat{i}+2\,\hat{j}+\hat{k}$ and $2\,\hat{i}-\hat{j}+\hat{k}$ are coplanar, find the value of X.

Section-B

3] Short answer type questions : [Each question caries 3 marks]

i) Find the direction cosines of the straight line joining the points (4, –3, –1) and (1, –1, 5).

ii) If projection of vector ($\lambda\,\hat{i}-\hat{j}$) $\hat{i}+\hat{j}$ on vector $\hat{i}-\hat{j}$ is zero, find λ.

iii) Prove that $\left(\vec{a}\times\vec{b}\right)^2 + \left(\vec{a}.\vec{b}\right)^2 = |\vec{a}|^2\,|\vec{b}|^2$

iv) If $\vec{a} = \hat{i}+2\,\hat{j}-\hat{k}$ and $\vec{b} = 3\,\hat{i}+\hat{j}-5\,\hat{k}$;find a unit vector in a direction parallel to vector $\left(\vec{a}-\vec{b}\right)$.

v) If $\vec{a} = 2\,\hat{i}+3\,\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-2\,\hat{j}+\hat{k}$ and $\vec{c} = -3\,\hat{i}+\hat{j}+2\,\hat{k}$, find $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

vi) Find a vector α of magnitude $5\sqrt{2}$, making an angle of $\pi/4$ with X-axis, $\pi/2$ with Y-axis and an acute angle θ with Z-axis.

vii) Find the angle between X-axis and the vector $\hat{i}+\hat{j}+3\hat{k}$.

viii) The midpoints of the sides of a triangle are (1, 5, –1), (0, 4, –2) and (2, 3, 4). Find the coordinates of its vertices of triangle.

ix) If α, β, γ be the direction angle of a straight line, then prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2$

x) If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be three unit vectors such that $|\vec{\alpha}+\vec{\beta}+\vec{\gamma}| = 1$ and $\vec{\alpha}$ is perpendicular to $\vec{\beta}$ while $\vec{\gamma}$ makes angles θ and ϕ with $\vec{\alpha}$ and $\vec{\beta}$ respectively, then find the value of $\cos\theta+\cos\phi$.

Section-C

4. Long answer type questions : [Each question caries 4 or 6 marks]

i) If $\vec{\alpha} = 3\hat{i}+4\,\hat{j}-\hat{k}$, $\vec{b} = \hat{i}-3\,\hat{j}+4\hat{k}$ and $\vec{c} = 5\,\hat{i}-6\,\hat{j}+4\,\hat{k}$, find the vector $\vec{r}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and which satisfies the relation $\vec{r}.\vec{c} = 91$.

ii) If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

iii) If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$, show that the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar.

iv) If $\vec{a}=3\hat{i}-\hat{j}$ and $\vec{b}=2\hat{i}+\hat{j}-3\hat{k}$, then express $\vec{b}$ in the form $\vec{b}=\vec{b}_1+\vec{b}_2$, where $\vec{b}_1 \parallel \vec{a}$ and $\vec{b}_2 \perp \vec{a}$.

v) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, then prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

vi) The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+3\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence, find the unit vector along $\vec{b}+\vec{c}$.

vii) A straight line makes angles α, β, γ, δ with the four diagonals of a cube; prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta={}^4\!/_3$

viii) If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ the position vectors of the point A, B, C, D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

ix) The position vectors of a point A is $\vec{a}+2\vec{b}$ and $\vec{a}$ divides AB internally in the ratio 2:3; then find the position vector of the point B.

x) Dot product of a vector with vectors $\hat{i}-\hat{j}+\hat{k}$, $2\hat{i}+\hat{j}-3\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$ are respectively 4, 0 and 2. Find the vector.

xi) The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.

xii) Prove that $\left[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{d}\right]=\left[\vec{a}\vec{b}\vec{d}\right]+\left[\vec{a}\vec{c}\vec{d}\right]$

xiii) Find the coordinates of the foot of the perpendicular drawn from the point A(1,8,4) on the straight line joining the point B(0,–11,4) and C(2,–3,1)

xiv) If θ be the angle between two unit vectors e_1 and e_2, prove that, $|\vec{e}_1-\vec{e}_2|=2\sin{}^{\theta}\!/_2$

xv) If a vector $\vec{v}$ is such that $2\vec{v}+\vec{v}\times\left[\hat{i}+2\hat{j}\right]=2\hat{i}+\hat{k}$ and $|\vec{v}|=\frac{1}{3}\sqrt{m}$, then find the value of m.

ANSWERS

Section-A

1)	i) d	ii) a	iii) a	iv) c	v) b	vi) a
	vii) c	viii) d	ix) d	x) b		

2) ii) $\hat{i}+\hat{j}$ iii) $\sqrt{69}$ and $\frac{1}{\sqrt{69}},\frac{-2}{\sqrt{69}},\frac{-8}{\sqrt{69}}$ v) ${}^{\pi}\!/_2$

vi) ${}^{\pi}\!/_4$ vii) –10

viii) a) $-2/3,\ 2/3,\ 1/3$ *or* $2/3, -2/3, -1/3$

b) $-6/7, -2/7, -3/7$ *or* $6/7,\ 2/7,\ 3/7$

ix) $\cos^{-1}\left(\frac{27}{2\sqrt{221}}\right)$ x) $x=\frac{29}{3}$

Section-B

3) i) $\frac{-3}{7}, \frac{2}{7}, \frac{6}{7}$ ii) $\lambda=1$ iv) $\frac{1}{\sqrt{21}}\left(-2\hat{i}+\hat{j}+4\hat{k}\right)$

v) -3 vi) $\vec{\alpha}=5\hat{i}+5\hat{k}$ vii) $\cos^{-1}\left(1/\sqrt{3}\right)$

viii) (27, –18, –54), (27, –18, 54), (–27, 18, –54) x) –1

Section-C

4). i) $13\left(\hat{i}-\hat{j}-\hat{k}\right)$

iv) $\vec{b}_1+\vec{b}_2=2\hat{i}+\hat{j}-3\hat{k}=\vec{b}$

v) $\cos^{-1}\left(1/\sqrt{3}\right)$

vi) $\lambda=1,\ \frac{3}{7}\hat{i}+\frac{6}{7}\hat{j}-\frac{2}{7}\hat{k}$

viii) π

ix) $\vec{a}-3\vec{b}$

x) $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$

xi) $2\sqrt{101}$ sq. units

xiii) (4, 5, –2)

xv) 6

CHAPTER- 11

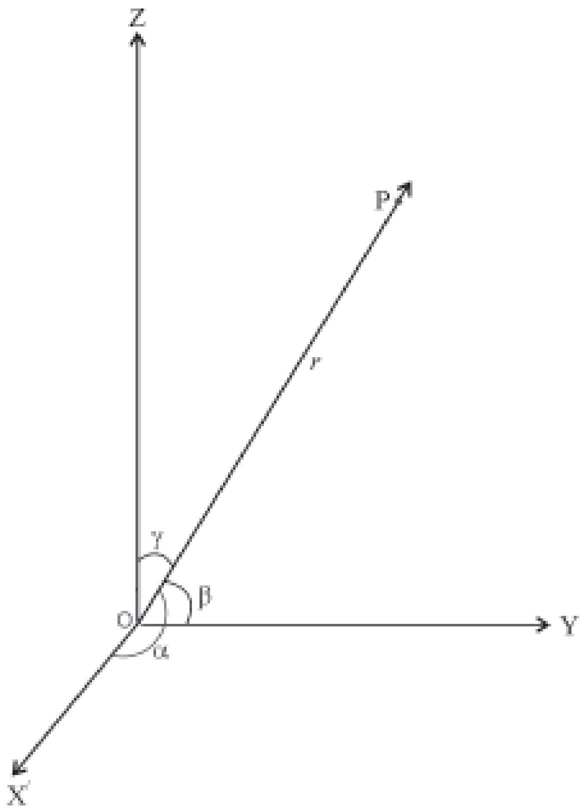
THREE DIMENSIONAL GEOMETRY

Important points and Results :

- **Direction cosines and direction ratios of a lines :**

  If a directed line $\overrightarrow{OP}$ passing through the orgin makes angles α, β and γ with x, y and z-axes respectively, called direction angles, then cosine of these angles, namely, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the directed line $\overrightarrow{OP}$.

  If we reverse the direction of $\overrightarrow{OP}$, then the direction angles are replaced by their supplements i.e $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$.



- **Direction ratios of a Lines :**
  Any three numbers which are proportional to the directoin cosines of a line are called the direction ratios of the line. If l, m, n are direction cosines and a, b, c are direction ratios of a lines then $a=\lambda l$, $b=\lambda m$ and $c=\lambda n$, for any nonzero $\lambda \in R$

- If l, m, n are direction cosines of a line, then $l^2+m^2+n^2=1$

- If *l, m, n* are the direction cosines and *a, b, c* are the direction ratios of a lines then $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$,

  $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$, $n = \frac{c}{\sqrt{a^2+b^2+c^2}}$.

- Direction cosines of a line joining two points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are $\frac{x_2-x_1}{PQ}$, $\frac{y_2-y_1}{PQ}$,

  $\frac{z_2-z_1}{PQ}$. When $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

- The directio ratios of the line segment joining $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ may be taken as x_2-x_1, y_2-y_1, z_2-z_1 or x_1-x_2, y_1-y_2, z_1-z_2.

- **Equation of a line in space :**
  A line is uniquely determined if
  i) it passes through a given point and has given direction.
  ii) it passes through two given points.

- Vector equaton of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$.

- Equation of a line through a point (x_1,y_1,z_1) and having direction cosines *l, m, n* is

  $$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

- Equation of a line passing through a point (x_1,y_1,z_1) and the direction ratio of the lin be *a, b, c* is

  $$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

  This is the cartesian equation of the line.

- The vector equation of a line which passes through two points whose positon vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\left(\vec{b}-\vec{a}\right)$, $\lambda \in R$.

- Cartesian equation of a line that passes through two points (x_1,y_1,z_1) and (x_2,y_2,z_2) is

  $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the acute angle between the two lines, then $\cos\theta = \left|\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$.

- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines, and θ is the acute angle between the two lines; then

  $\cos\theta = | \, l_1l_2 + m_1m_2 + n_1n_2 \, |$ and

  $\sin\theta = \sqrt{(l_1m_1 - l_2m_2) \;\; - (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2}$

- If θ is the acute angle between $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, then $\cos\theta = \left|\dfrac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$.

- If $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ are the equation of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$.

- Two lines with direction rations a_1, b_1, c_1 and a_2, b_2, c_2 are
  i) Perpendicular i.e. $\theta = 90^0$ then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
  ii) Parallel i.e. if $\theta = 0$ then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

- **Skew lines** are lines in space which are neither parallel nor intersecting. They lie in different planes.

- **Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.

- The shortest distance between two lines we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

  For skew lines, the line of the shortest distance will be perpendicular to both the lines.

- Shortest distance two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $\left|\dfrac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

- Shortest distance between two skew lines $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$

is $\dfrac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$.

- Distance between two parallel lines $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ is $\left|\dfrac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$.

- A plane is determined uniquely if any one, of the following is known :
  i) The normal to the plane and its distance from the origin is given i.e. equation of a plane in normal form.
  ii) It passes through a point and is perpendicular to a given direction.
  iii) It passes through three given non-collinear points.

- In the vector form, equation of a plane which is at a distance *d* from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is $\vec{r}.\hat{n}=d$.

- Equation of a plane which is at a distance *d* from the origin and the direction cosines of the normal to the plane as *l, m, n* is $lx+my+nz=d$

- If $\vec{r}.\left(a\hat{i}+b\hat{j}+c\hat{k}\right)=d$ is the vector equation of a plane, then $ax+by+cz=d$ is the cartesian equation of the plane, where *a, b* and *c* are the direction ratios of the normal to the plane.

- If *d* is the distance from the origin and *l, m, n* are the direction cosines of the normal to the plane through the origin, then the foot of the perpendicular is (*ld, md, nd*).

- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r}-\vec{a}).\vec{N}=0$.

- Equation of a plane perpendicular to a given line with direction ratios *A, B, C* and passing through a given point (x_1, y_1, z_1) is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$.

- Vector equation of a plane that contains three non-collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $(\vec{r}-\vec{a}).\left[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right]$.

- Equation of a plane passing through three non collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

- Equation of a plane that cuts the co-ordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

- Vector equation of a plane that passes through the intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$, where λ is any non-zero constant.

- Cartesian equation of a plane that passes through the intersection of two given planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ is $(A_1x+B_1y+C_1z+D_1)+\lambda(A_2x+B_2y+C_2z+D_2)=0$

- Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are coplanar, if $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$.

- In the cartesian form two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplaner

if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- **Angle between two planes :**
The angle between two planes is defined as the angle between their normals. Observe that if θ is an angle between the two planes, then so is 180–θ. We shall take the acute angle as the angles between two planes.

If $\vec{n}_1$ and $\vec{n}_2$ are normals to the planes and θ be the angle between the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$. Then θ is the angle between the normals to the planes drawn from some common point.

We have, $\cos\theta = \left|\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$.

- The planes are perpendicular to each other if $\vec{n}_1.\vec{n}_2 = 0$ and parallel if $\vec{n}_1$ is parallel to $\vec{n}_2$.

- The angle θ between the planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ is given by

$$\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|.$$

- If the planes are at right angles, then θ=90° and so cosθ=0. Hence $\cos\theta = A_1A_2+B_1B_2+C_1C_2=0$.

- If the planes are parallel, then $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.

- The angle ϕ between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}.\hat{n} = d$ is $\sin\varphi = \left|\dfrac{\vec{b}.\hat{n}}{|\vec{b}||\hat{n}|}\right|$.

- The distance of a point whose position vector is $\vec{a}$ from the planes $\vec{r}.\hat{n} = d$ is $|d - \vec{a}.\hat{n}|$.

- The distance from a point (x_1, y_1, z_1) to the plane $Ax+By+Cz+D=0$ is $\left|\dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$.

Exercise- 11
Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) Multiple choice type questions :

i) The value of m for which the planes $2x+3y-z=5$ and $3x-my+3z=6$ are perpendicular to each other is

a) -1 b) $1/2$ c) 1 d) $-1/2$

ii) Find the value of λ if three vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.

a) -3 b) 3 c) -4 d) 4

iii) The line $\dfrac{x-1}{2} = \dfrac{y-2}{-3} = \dfrac{z+5}{4}$ meets the plane $2x+4y-z=3$ at the point whose coordinates are

a) $(3, 1, -1)$ b) $(3, -1, 1)$ c) $(3, -1, -1)$ d) none of these

iv) Find the angle between the planes $x+y+2z=6$ and $2x-y+z=9$ is

a) $\pi/4$ b) $\pi/6$ c) $\pi/2$ d) $\pi/3$

v) The distance of the point $(1, 0, 2)$ from the point of intersection of the line $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z-2}{12}$ and the plane $x-y+z=16$ is

a) $3\sqrt{21}$ b) 13 c) $2\sqrt{14}$ d) 8

vi) If the shortest distance between the lines $\dfrac{x-1}{1} = \dfrac{y-1}{1} = \dfrac{z-1}{1}$ and $\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z-4}{1}$ is equal to $\sqrt{K}$ unit, then the values of K is –

a) 3 b) 4 c) 2 d) 5

vii) The foot of the perpendicular from the point (1, 1, 2) to the plane $2x-2y+4z+5=0$ is –

a) $\left(\frac{1}{12},\frac{25}{12},\frac{2}{12}\right)$ b) $\left(-\frac{1}{12},\frac{25}{12},\frac{-2}{12}\right)$ c) $\left(\frac{1}{12},\frac{25}{12},\frac{-2}{12}\right)$ d) none of these

viii) The equation of the image of the line $\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3}$ in the plane $3x-3y+10z-26=0$ is

a) $\frac{2x-5}{18}=\frac{2y-1}{2}=\frac{z-2}{3}$ b) $\frac{2x-5}{18}=\frac{2y+1}{-2}=\frac{z-2}{3}$

c) $\frac{2x+5}{18}=\frac{2y-1}{2}=\frac{z-2}{-3}$ d) $\frac{2x-5}{18}=\frac{2y-1}{-2}=\frac{z-2}{-3}$

ix) The angle between the plane $\vec{r}.\left(\hat{i}-3\hat{j}+2\hat{k}\right)=15$ and the line.

$\vec{r}=\left(\hat{i}+\hat{j}+\hat{k}\right)+s\left(2\hat{i}+\hat{j}-3\hat{k}\right)$ is θ, then the value of cosecθ is –

a) 5 b) 2 c) 4 d) 3

x) The condition for collinearity for the two parallel lines $\vec{r}=\vec{a}+t\vec{b}$ and $\vec{r}=\vec{a}_1+s\vec{b}$ is

a) $\left|(\vec{a}-\vec{a}_1)\cdot\vec{b}\right|=0$ b) $\left|(\vec{a}+\vec{a}_1)\cdot\vec{b}\right|=0$

c) $\left|(\vec{a}+\vec{a}_1)\times\vec{b}\right|=0$ d) $\left|(\vec{a}-\vec{a}_1)\times\vec{b}\right|=0$

2] Very short answer type questions : [Marks each questions 1 or 2]

i) Write the direction cosines of the lines $6x-2=3y+1=2z-4$.

ii) Find the equation of a line passing through the point (1, 2, 3) and parallel to the line $\frac{x-1}{2}=\frac{7-y}{3}=-z$.

iii) Find the perpendicular distance of the point (1,1,0) from the Z-axis?

iv) Find the equation of the plane which makes equal (non-zero) intercepts on the axes and passes through the point (2, 3, –1).

v) Find the equation of the plane which passes through the point (2, 1, –1) and is orthogonal to each the planes $x-y+z=1$ and $3x+4y-2z=0$.

vi) Find the equation of the plane passing through the points (0,0,0) and (3,–1,2) and parallel to the line $\frac{x-4}{1}=\frac{y+3}{-1}=\frac{z+1}{7}$.

vii) The ratio in which the plane $\vec{r}.\left(\hat{i}-2\hat{j}+3\hat{k}\right)=17$ divides the line joing the points $-2\hat{i}+4\hat{j}+7\hat{k}$ and $3\hat{i}-5\hat{j}+8\hat{k}$ is –

a) 1:5 b) 1:10 c) 3:5 d) 3:10

viii) Obtain the equation of the plane passing through the point (1,–3,–2) and perpendicular to the planes $x+2y+2z=5$ and $3x+3y+2z=8$.

ix) Find the distance of the point (1,–5,9) from the plane $x-y+z=5$ measured along the line $x=y=z$.

x) If the line $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane $lx+my-z=9$ then show that l^2+m^2 is equal to 2.

Section-B

3) Short answer type questions : [Each quetion caries 3 marks]

i) Two lines $l_1: x=5, \frac{y}{3-\alpha}=\frac{z}{-2}$ and $l_2: x=\alpha, \frac{y}{-1}=\frac{z}{2-\alpha}$ are coplaner. Then find the value of α.

ii) Find the angle between pair of lines $\vec{r}=\left(2\hat{i}+3\hat{j}-5\hat{k}\right)+t\left(\frac{1}{2}\hat{i}+\hat{j}-\hat{k}\right)$ and $\vec{r}=\left(2\hat{i}-6\hat{k}\right)+t'\left(\hat{i}+2\hat{j}-2\hat{k}\right)$.

iii) Show that the lines $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ and $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ do not intersect.

iv) Show that the lines $\vec{r}=\hat{i}+t\left(5\hat{i}+2\hat{j}+\hat{k}\right)$ and $\vec{r}=\hat{i}+s\left(-10\hat{i}-4\hat{j}-2\hat{k}\right)$ are coincident.

v) Find the equation of the plane which passes through the points (1,2,3), (2,3,1) and (3,1,2).

vi) Show that $\left(-\frac{1}{2},2,0\right)$ is the circum-centre of the the triangle formed by the points (1,2,1), (–2,2,–1) and (1,1,0).

vii) Find the angles between the plane $x+8y-6z+16=0$ and the co-ordinate planes.

viii) Find the equation of the plane parallel to the plane $2x-2y-z-3=0$ and situated at a distance of 7 units from it.

ix) If the cartesian equation of a line AB is $\frac{x-1}{2}=\frac{2y-1}{12}=\frac{z+5}{3}$, then find the direction cosines of a line parallel to AB.

x) Find the vector equation of the plane passing through the point (1,–1,2) and having 2, 3, 2 as direction numbers of the normal to the plane.

Section-C

4. Long answer type questions : [Each quetion caries 3 marks]

i) Find the image of the point (1, 6, 3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$. Also, write the equation of the line joining the given point and its image and find the length of the segment joing the given point and its image.

ii) Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x}{2}=\frac{y-5}{3}=\frac{z+1}{4}$.

iii) Find the image of the point with position vector $\left(3\hat{i}+\hat{j}+2\hat{k}\right)$ in the plane $\vec{r}.\left(2\hat{i}-\hat{j}+\hat{k}\right)=4$.

iv) Find the distance of the point (1, –2, 3) from the plane $x-y+z=5$ measured along a line parallel to $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$.

v) Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.\left(2\hat{i}-3\hat{j}+4\hat{k}\right)=1$ and $\vec{r}.\left(\hat{i}-\hat{j}\right)+4=0$ which is perpendicular to $\vec{r}.\left(2\hat{i}-\hat{j}+\hat{k}\right)+8=0$

vi) Find the question of line through (1, 2, 4), which is paralle to the planes $3x+2y-z-4=0$ and $x-2y-2z-5=0$.

vii) Find the equation of the plane containing the lines $\vec{r}=\hat{i}+\hat{j}+\lambda(\hat{i}+2\hat{j}-\hat{k})$ and $\vec{r}=\hat{i}+\hat{j}+\mu(\hat{i}+\hat{j}-2\hat{k})$. Find the distance of this plane from the origine and also from the point (2, 2, 2).

viii) Find the image of line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3=0$.

ix) Find the equation of the plane that contains the point (1, –1, 2) and is perpendicular to both the planes $2x+3y-2z=5$ and $x+2y-3z=8$. Hence, find the distance of point $P(-2, 5, 5)$ from the plane obtained above.

x) Let P be the image of the point (3, 1, 7) with respect to the plane $x-y+z=3$. Then, the find the equation of the plane passing through P and containing the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$.

ANSWERS

Section-A

1) i) c ii) c iii) c iv) d v) b vi) c
vii) b viii) d ix) b x) b

2) i) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ ii) $\frac{x-11}{2} = \frac{2-y}{3} = \frac{3-z}{1}$

iii) 4.18 units iv) $x+y+z = 4$

v) $-2(x-2)+5(y-1)+7(z+1) = 0$ vi) $x-19y-11z = 0$

vii) 766; 3:10 viii) $2x-4y+3z-8 = 0$

ix) $10\sqrt{3}$ unit

Section-B

3) i) 4, 1 ii) 0 v) $x+y+z=6$

vii) xy plane : $\cos^{-1}\left(\frac{-6}{\sqrt{101}}\right)$; yz plane : $\cos^{-1}\left(\frac{1}{\sqrt{101}}\right)$; zx plane : $\cos^{-1}\left(\frac{8}{\sqrt{101}}\right)$

viii) $2x-2y-z = 24$ and $2x-2y-z+18 = 0$

ix) $2/7, 6/7, 3/7$ x) $\vec{r}.\left(2\hat{i}+3\hat{j}+2\hat{k}\right) = 3$

Section-C

4) i) (1,0,7) and $2\sqrt{13}$ unit ii) 4.817 unit iii) $\left(\hat{i}+\hat{j}+\hat{k}\right)$

iv) 1 v) $\vec{r}.\left(-5\hat{i}-2\hat{j}+12\hat{k}\right) = 47$ vi) $\frac{x-1}{6} = \frac{y-2}{-5} = \frac{z-4}{8}$

vii) $-x+y+z = 0$, $\frac{2}{\sqrt{3}}$ unit viii) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$ ix) $5x-4y-z-7=0$ and $\sqrt{42}$ unit

x) $x-4y+7z = 0$

CHAPTER- 12

LINEAR PROGRAMMING PROBLEM

Important points and Results :

- **Linear Progamming :**

  Linear programming is a powerful mathematical technique for determing the best possible (optimal) solution in allocating limited resources (energy, machines, meterials, money, personnel, space, time etc) to achieve maximum profit or minimum cost.

- **Linear Programming Problem :**

  A linear programming problem is an optimizatoin problem where

  i) We attempt to maximize (or minimize) a linear function of decision variables (objective function).

  ii) The values of the decision veriables must satisfy a set of constranints, each of which is a linear equation or inequation.

  iii) For each decision variable x_j, the sign of of restriction is $x_j \geq 0$.

  A linear programming problem is an optimization problem that is concerned with the finding of the optimal (maximum or minimum) value of linear function (called objective function) of two (or more than two) real variables which are non-negative and satisfy a set of linear equalities and inequalities (called constraints). The variables are called decision variables.

  Now we formally define some important terms related to linear programming problem.

  i) Objective function :
  Linear function $Z = ax+by$, where a, b are constants, which has to be maximized or minimized is called a linear objective function.

  ii) Decision variables :
  If $Z=ax+by$ is a linear objective function, then x and y are called decision variables.

  iii) Constraints :
  The linear inequalities or equations or restrictions on the variables of a linear programming problem are called constraints. The conditions $x\geq 0$, $y\geq 0$ are called non-negetive restrictions.

  iv) Optimisation Problem :
  A problem which seeks to maximise or minimise a linear function (Say of two variables x and y) subject to certain constraints as determined by a set of linear inequalities is called an optimisation problem. LPP are special type of optimisation problems.

- **Different types of Linear programming problems :**

  A few important linear programming problems are listed belew :

  1) Manufacturing Problems : In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hour per unit of product, warehouse space per unit of the output etc., in order to make maximum profit.

2) **Diet Problems :** In these problems, we determine the amount of different kinds of constitutents/ nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each constituents/nutrients.

3) **Transportation Problems :** In these Problem, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situate at different locations to different markets.

The common region determined by all the constraints including the non negative constraints $x \geq 0$, $y \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem.

- Points with in and on the boundary of the feasible region represent feasible solutions of the constraints.
- Any point outside the feasible region is an infeasible solution.
- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.
- If the feasible is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R.
- Corner Point method for solving a linear programming problem. The method comprises of the following steps –
  - i) Find the feasible region of the linear programming problem and determine its corner points (vertices)
  - ii) Evaluate the objective function $Z=ax+by$ at each corner point. Let M and m respectively are the maximum and minimum values of the objective function.
- If the feasible region is unbounded, then
  - i) M is the maximum value of the objective function, if the open half plane determined by $ax+by>M$ has not point in common with the feasible region. Otherwise, the objective function has no maximum value.
  - ii) m is the minimum value of the objective function, if the open half plane determined by $ax+by<m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.
- If two corner points of the feasible region are both optimal solutions of the same type i.e. both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.
- An L.P.P with two decision variable can be solved graphically. The corner point (points) and hence the solution of a L.P.P with two decision variables can be identified graphically.
- A linear programming problem may have a unique and finite optimal solution. In some problems, there may be more than one solution. It is also possible that a linear programming problem has no finite solution (unbounded solution). Sometimes a linear programming problem may have no feasible solution.

Excercise- 12
Section-A

A] OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) Multiple choice type questions : (Choose the correct option)

i) The objective function of an LPP is

a) irrational function of decesion variables.

b) trigonometric function of decision variables.

c) exponential function of decision variables.

d) linear function of decision variables.

ii) Given the LPP, Max $Z=6x+10y$

Subject to the constraints $3x+5y \le 10,$

$5x+3y \le 15$

and $x, y \ge 0$.

The number of optimal solutions of the LPP is

a) one b) two c) finite d) infinite

iii) Any solution to an L.P.P. which satisfies the non-negative restrictions of the problem, is called

a) optimal solution b) feasible solution c) basic solution d) none of these.

iv) In an LPP, the decision variables can take

a) any real values b) integer values only

c) any non-negative real values d) none of these

v) An infeasible linear programming problem has

a) a unique solution b) many solutions

c) two distinct solutions d) no solution

vi) If the value of the objective function of an LPP can be increased or decreased indefinitely, then the LPP is said to have

a) an unbounded solution b) an infinite solution

c) a bounded solution d) no solution

vii) An unbounded solution of a linear programming problem is a solution whose objective function is

a) zero b) a large positive real number

c) a large negative real number d) infinite

viii) Which of the following is true in an LPP ?

a) Min.Z = – Max (–Z) b) Min.Z = – Max.Z

c) Min.Z = Max(–Z) d) none of these

ix) In an LPP, the equation $2x+3y=12$ in two unknown has number of solutions equal to –

a) a particular value of x and y b) maximum value of x and minimum value of y

c) infinite d) none of these

x) Given the LPP, Max. Z = $3x+4y$

subject to the constraints $-2x+3y \le 9$

$x-5y \ge -20$,

and $x, y \ge 0$

The LPP has –

a) a unique optimal solution b) alternative optional solutions

c) an unbounded solution d) none of these

xi) Given the LPP, Min.Z=$3x-y$

Subject to the constraints $2x+3y \ge 1$,

and $x, y \ge 0$

The optimal solution of the LPP is –

a) $x=0,\ y=\frac{1}{2}$ b) $x=0,\ y=\frac{1}{3}$ c) $x=\frac{1}{3},\ y=0$ d) $x=\frac{1}{2},\ y=0$

xii) Given the LPP, Max.Z = $x+y$

Subject to the constrains $x+2y \le 4$,

$x+2y \ge 6$

and $x, y \ge 0$

The given LPP has –

a) unique feasible solution b) infinite number of feasible solution

c) no feasible solution d) none of these

xiii) An unbounded feasible region

a) admits bounded feasible solution b) admits unbounded solution

c) may admit bounded as well as unbounded feasible solution

d) none of these

xiv) State which of the following statement is false ?

a) The most important feature of an LPP is the presence of linearity in the problem.

b) The objective function may assume its optimal value at more than one corner point of the feasible region.

c) Multiple solutions of linear programming problem are solutions each of which maximize or minimize the objective function.

d) An infeasible LPP has no feasible solution.

2] Very short answer type questions :

i) What is Linear programming? Explain with an example.

ii) Define Linear programming problem.

iii) Write down two advantages of an LPP.

iv) Define feasible solution and feasible region.

v) Define objective function.

vi) Define decision variables.

vii) What is constraints ?

viii) What do you mean by optimization problem ?

ix) Define optimal solution.

x) Explain extreme or corner point.

Section-B

3] **Short answer type questions :** [3 marks for each question]

i) Given the LPP, Max.Z = $2x+3y$
Subject to the constraints
$$3x+y \leq 3,\ \ x \geq 0,\ \ y \geq 0$$
Show that the corner points of the LPP are (0, 0), (1, 0) and (0, 3).

ii) Find the optimal solution of the above LPP and the maximum value of Z.

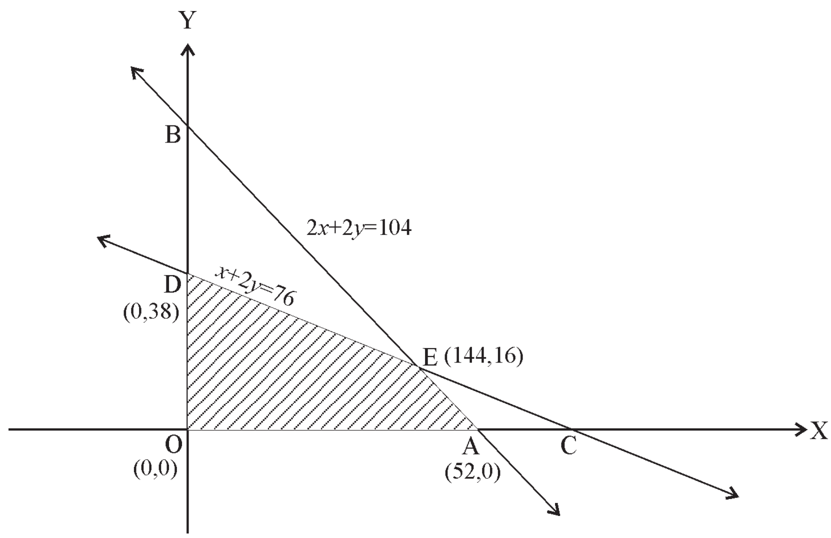
iii) Given the LPP, Max.Z = $3x_1+2x_2$
Subject to the constraints $2x_1+x_2 \leq 2$
$3x_1+4x_2 \geq 12$
$x_1 \geq 0,\ x_2 \geq 0$
Draw the graph of the constraints.

iv) Find graphically the feasible region, if any for the following inequations
$x \leq 2,\ y \leq 3,\ x + y \geq 1$ and $x,\ y \geq 0$

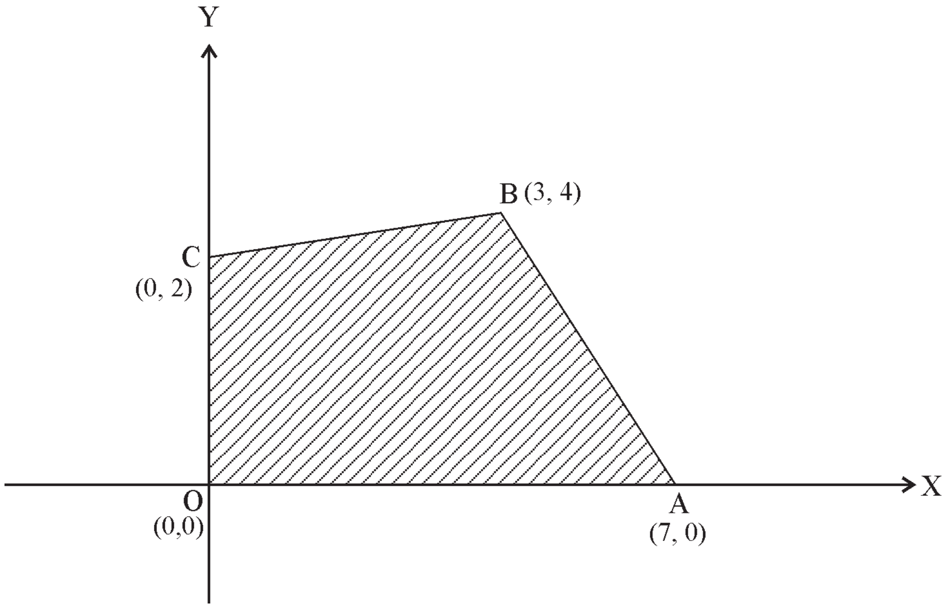
v) Find the corner points of the LPP, Max.Z = $x+2y$
Subject to the constraints $3x+5y \leq 10$
$5x+3y \leq 15$
$x,\ y \geq 0$

vi) Find the corner points of the LPP, Max.Z=$2x+5y$
Subject to the constraints $0 \leq x \leq 4$
$0 \leq y \leq 3$
$x+y \leq 6$

vii) Determine the maximum value of Z=$3x+4y$, if the feasible region (shaded) for an LPP in shown in the following figure.



viii) Feasible region (shaded) for a LPP is shown in following figure. Find the maximam value of Z=$5x+7y$.



Section-C

Long answer type questions : (Each question carries 4 or 6 marks)

1) i) Solve the following LPP graphically

$$\text{Max } Z = 2x_1 + x_2$$

Subject to the constraints $x_1 + 3x_2 \leq 15$

$3x_1 - 4x_2 \leq 12$

$x_1 \geq 0, \ x_2 \geq 0$

Find the corner points of the convex set of feasible solution.

ii) Solve the following LPP graphically.

$$\text{Min } Z = 5x + 7y$$

Subject to the constraints $3x + 2y \geq 12$

$2x + 3y \geq 13$

and $x \geq 0, y \geq 0$

2) A furniture manufacturing company plans to make two products, chairs and tables from its available resources of 400 board feet of mahogany timber and 450 man-hours. A chair requirs 5 board feet of timber and 10 man-hours and yields a profit of ₹45, while each table uses 20 board feet of timber and 15 man-hours and a profit of ₹80, formulate the problem as an LPP to maximize profit.

$$\left[1 \; board \;\; feet = \frac{1}{12} cubic \;\; feet\right]$$

3) Make a graphical representation of the set of constraints in the following LPP –

$$\text{Maximize } Z = x + y$$

Subject to the constraints $5x + 10y \leq 50$

$x + y \geq 1$

$y \leq 4$

and $x \geq 0, y \geq 0$

Show that the LPP has unique optimal solution.

4) Food F_1 contains 5 units of vitamin A and 6 units of vitamin B per gram and costs 20p/gm. Food F_2 contains 8 units of vitamin A and 10 units of vitamin B per gram and costs 30 p/gm. The daily requirements of A and B are at least 80 and 100 units respectively. Formulate the problem as a linear programming problem to minimize cost.

5) Given the LPP, Max $Z = 2x+3y$

Subject to the constraints $3x - y \leq -3$

$x - 2y \geq 2$

and $x \geq 0, y \geq 0$

Graphically show that the LPP has no feasible solution.

6) There are two factories located one at place A and the other at B. From these locations, a certain commodity is to be delivered to each of the three depots situated at C, D, E. The weekly requirements of the depots are reseptively 5, 5 and 4 units of the commodity while the production capacity of the factories at A and B are respectively 8 and 6 units. The cost of transportation per unit is given below:

To		Cost in R		
		C	D	E
From	A	R 160	R 100	R 150
	B	R 100	R 120	R 100

How many units should be transported from each factory to each depot in order to minimize the transportation cost. Pose the problem as an LPP.

7) Show by graphical method that the feasible region of the following LPP is unbounded but it has unique optimal solution $x=3, y=18$.

Minimize $Z = 4x + 2y$

Subject to the constraints $3x + y \geq 27$

$-x - y \leq -21$

$x + 2y \geq 30$

and $x \geq 0, \ y \geq 0$

8) Show graphically that the following LPP has an unbounded solution

Maximize $Z = 3x + 4y$

Subject to the constraints $-2x + 3y \leq 9$

$x - 5y \geq -20$

and $x \geq 0, \ y \geq 0$

9) A company manufactures two types of toys A and B. Type A requires 5 minutes each for cutting and 10 minutes each for assembling.

Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours available for cutting and 4 hours available for assembling in a day. He earns a proft of R50 each on type A and R60 each on type B. How many toys of each type should the company manufacture in a day to maximize the profit ?

10) A house wife wishes to mix together two kinds of food, X and Y, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of 1 kg of each food are given below :

	Vitamin A	Vitamin B	Vitamin C
Food X	1	2	3
Food Y	2	2	1

If 1 kg of food X costs R6 and 1 kig of food Y costs R10, find the minimum cost of the mixture which will produce the diet.

11) A transport company has offices in five localities A, B, C, D and E. Some day the offices located at A and B has 8 and 10 spare trucks whereas offices at C, D, E required 6, 8 and 4 trucks respectively. The distance in kilometres between the five locations are given below :

To	C	D	E
A	2	5	3
B	4	2	7

How should the trucks from A and B be sent to C, D and E so that the total distance covered by the trucks is minimum. Formulate the problem as an LPP and solve it graphically.

12) A firm manufactures two types of products, A and B, and sells them at a a profit of R5 per unit of type A and R3 per unit to type B. Each product is processed on two machines, M_1 and M_2. One unit of type A requires one minute of processing time on M_1 and two minutes of processing time on M_2; Whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machines M_1 and M_2 are respectively available for at most 5 hours and 6 hours in a day. Find out how many units of each type of product the firm should produce a day in order to maximize the profit. Solve the problem graphically.

13) Solve the following LPP Graphically :

$$\begin{aligned} \text{Maximise } & Z = x + y \\ \text{Subject to } & x + 4y \le 8, \\ & 2x + 3y \le 12 \\ & 3x + y \le 9, \\ \text{and } & x \ge 0,\ y \ge 0 \end{aligned}$$

14) Solving graphically, show that the following LPP has an infinite number of optimal solutions.

$$\begin{aligned} \text{Minimize } & Z = x + y \\ \text{Subject to the constraints } & 5x + 9y \le 45 \\ & x + y \ge 2 \\ & x \le 4 \\ \text{and } & x \ge 0,\ y \ge 0 \end{aligned}$$

Find also the minimum value of the objective function Z.

15) Solve the following LPP by the graphical method

$$\text{Min. } Z = 3x + y$$

Subject to the constraints $2x + y \geq 14$

$x - y \geq 4$

and $x \geq 0,\ y \geq 0$

ANSWERS

Section-A

1) Multiple Choice type questions : (Choose the correct answer)

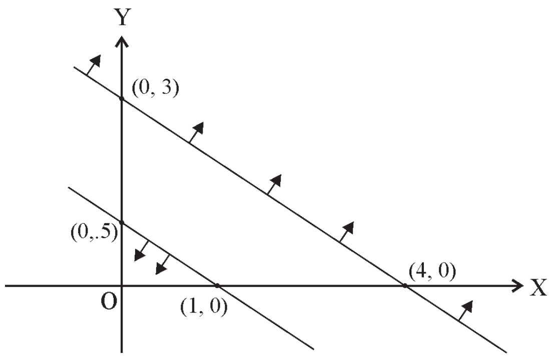
i) d	ii) a	iii) b	iv) c	v) d	vi) a
vii) d	viii) a	ix) c	x) c	xi) b	xii) c
xiii) c	xiv) c				

Section-B

3) Short Answer type question :

ii) (0, 3), 9

iii)



v) (0, 0) (3, 0) (0, 2) $\left(\frac{45}{16}, \frac{5}{16}\right)$

vi) (0, 0) (4, 0) (4, 2) (3, 3) (0, 3)

vii) (44, 16), 196

viii) 43 at (3, 4)

Section - C

1) i) (0, 5) $\left(\frac{96}{13}, \frac{33}{13}\right)$, (4, 0) and (0, 0)

ii) (2, 3) and $Z_{max}=31$

2) Max $Z = 45x + 80y$

Subject to the constraints $5x + 20y \leq 400$

$10x + 15y \leq 450$

and $x \geq 0,\ y \geq 0$

Where x and y are the number of chairs and tables respectively.

4) Min $Z = 20x + 30y$

Subject to the constraints $5x + 8y \geq 80$

$6x + 10y \geq 100$

and $x \geq 0,\ y \geq 0$

Where x gms food F_1 and y gms of food F_2 are purchased.

6) Min $Z = 10(x - 7y + 190)$

Subject to the constraints $x + y \leq 8$

$x \leq 5, y \leq 5,$

$x + y \geq 4$

and $x \geq 0,\ y \geq 0$

Where x units and y units of the commodity are transported from the factory A to the depots at C and D respectively.

9) 12 toys of type A and 15 toys of type B to earn maximum proft ₹1500.

10) ₹52

11) The optimal solution is given by $x = 4, y = 0$ and $Z_{min} = 44$

12) Z is maximum at P(60, 240) and its maximum value is ₹1020.

13) The maximum value is $3\frac{10}{11}$.

14) $Z_{min} = 2$

15) $Z_{min} = 20$ and $x = 6, y = 2$.

PROBABILITY

Important points and Results :

- **Conditional probability :**

  If E and F are two events associated with the sample space of a random experiment, then the conditional probability of the event E under the conditon that the event F has already occured and P(F)≠0 is called the condition probability and it is denoted by $P\left(E/F\right)$.

  Thus, we have, $P\left(E/F\right)=\dfrac{P(E\cap F)}{P(F)}=\dfrac{n(E\cap F)}{n(F)}$.

  Similarly, $P\left(F/E\right)$ when P(E) $\neq 0$ is defined as the probability of occurance of event F when E has already occured.

  Infact, the meanings of symbols $P\left(F/E\right)$ and $P\left(F/E\right)$ depend on the nature of the events E&F and also on the nature of the random experiment.

- **Properties :**

  i) Let E and F be two events associated with a sample space S, then $0\leq P\left(E/F\right)\leq 1$.

  ii) If E and F be events associated with the sample space of a random experiment, then

  $$P\left(S/F\right)=P\left(F/F\right)=1$$

  $$P\left(E'/F\right)=1-P\left(E/F\right)$$

  iii) If E and F be two events associated with a random experiment and S be the sample space. If G is an event such that P(G)≠0, then

  $$P\left((E\cup F)/G\right)=P\left(E/G\right)+P\left(F/G\right)-P\left((E\cap F)/G\right)$$

  In particular, if E and F are mutually exclusive events, then

  $$P\left((E\cup F)/G\right)=P\left(E/G\right)+P\left(F/G\right)$$

- **Multiplication Theorem on probability :**

  Let E and F be two events associated with a sample space of an experiment. Then

  $$P(E\cap F)=P(E).P\left(F/E\right),\ P(E)\neq 0$$

  or, $$P(E\cap F)=P(F).P\left(E/F\right),\ P(F)\neq 0$$

If E, F and G are three events associated with a sample space, then

$$P(E \cap F \cap G) = P(E).P\left(F/_E\right).P\left(G/_{E \cap F}\right)$$

Note : $P\left(F/_E\right) = \dfrac{P(E \cap F)}{P(E)}$ and $P\left(E/_F\right) = \dfrac{P(E \cap F)}{P(F)}$

- **Extension of multiplication theorem :**

  If $E_1, E_2, ..., E_n$ are n events associated with a random experiment, then

  $$P(E_1 \cap E_2 \cap .... \cap E_n) = P(E_1).P\left(E_2/_{E_1}\right)P\left(E_3/_{E_1 \cap E_2}\right)..............P\left(E_n/_{E_1 \cap E_2 \cap .... \cap E_{n-1}}\right),$$

  where $P\left(E_i/_{E_1 \cap E_2 \quad ...... \cap E_{i-1}}\right)$ represents the conditional probability of the occurance of event E_i, given that the events $E_1, E_2, ..., E_{i-1}$ have already occured.

- **Independent Events :**

  Let E and F be two events associated with a sample space S. If the probability of occurance of one of them is not affected by the occurance of the other, then we say that the two events are independent. Thus, two events E and F will be independent, if

  $P\left(F/_E\right) = P(F)$, Povided $P(E) \neq 0$

  $P\left(E/_F\right) = P(E)$, provided $P(F) \neq 0$

  For example : Suppose a bag contains 6 white and 3 red balls. Two balls are drawn from the bag one after the other.

  We consider the following events :

  E = drawing a white ball in first draw.

  F = drawing a red ball in second draw.

  If the ball drawn in the first draw is not replaced back in the bag, then events E and F are dependent events. On the other hand, if the ball drawn in first draw is replaced back in the bag, then E and F are independent events.

- If E and F are independent events associated with a random experiment, then $P(E \cap F) = P(E).P(F)$

  Three events E, F and G are said to be mutually independent if all the following conditions hold :

  $P(E \cap F) = P(E)\, P(F)$

  $P(E \cap G) = P(E)\, P(G)$

  $P(F \cap G) = P(F)\, P(G)$

  and $P(E \cap F \cap G) = P(E)P(F)P(G)$.

- If $E_1, E_2, ..., E_n$ be n events associated to a random experiment, then these events are said to be mutually independent if the probability of the simultaneous occurance of any finite number of them is equal to the product of their separate probabilities i.e.,

$$P(E_i \cap E_j) = P(E_i)P(E_j), \text{ for } i \neq j;\ i, j=1, 2, ...n$$

$$P(E_i \cap E_j \cap E_k) = P(E_i)P(E_j)P(E_k), \text{ for } i \neq j \neq k;\ i, j, k = 1, 2, ...n.$$

$$\vdots \qquad\qquad \vdots$$

$$P(E_1 \cap E_2 \cap ....\cap E_n) = P(E_1)P(E_2)..........P(E_n)$$

If $E_1, E_2, ..., E_n$ are independent events associated with a random experiment, then

$P(E_1 \cap E_2 \cap ....\cap E_n) = P(E_1)P(E_2)......P(E_n)$

- **Pairwise independent events :**

  If $E_1, E_2, ..., E_n$ be n events associated to a random experiment, then these events are said to be pairwise independent if $P(E_i \cap E_j) = P(E_i)P(E_j)$, for $i \neq j$; $i, j=1,2,......, n$.

- **Remark :**

  i) If $E_1, E_2, ...E_n$ are pairwise independent events, then the total number of conditions for their pairwise independence is ${}^nc_2$ whereas for their mutual independencies there must be ${}^nc_2 + {}^nc_3 + ... + {}^nc_n = 2^n - n - 1$ conditions.

  ii) Mutually independent events are always pairwise independent but the converse need not be true.

  iii) In case of two events only associated to a random experiment, there is no distinction between their mutual independence and pairwise independence

- If A and B are independent events associated with a random experiment, then

  i) $\overline{A}$ and B are independent events.

  ii) A and $\overline{B}$ are independent events.

  iii) $\overline{A}$ and $\overline{B}$ are also independent events.

- **Remark :**

  i) Independent events will mean mutually independent events.

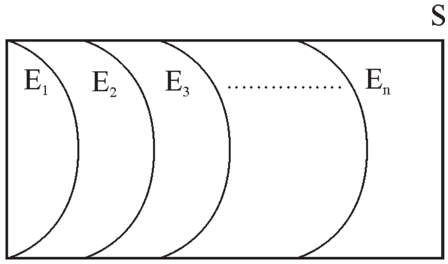
  ii) If A and B are independent events associated to a random experiment, then probability of occurance at least one $= P(A \cup B) = 1 - P(\overline{A})P(\overline{B})$

  iii) For n events $A_1, A_2, ..., A_n$ associated with a random experiment, then probability of occurance of at least one $= P(A_1 \cup A_2 \cup ....\cup A_n) = 1 - P(\overline{A_1})P(\overline{A_2})..........P(\overline{A_n})$

- **Independent experiments :**

  Two random experiments are independent if for every pair of events A and B where A is associated with the first experiment and B with the second experiment, we have $P(A \cap B) = P(A)P(B)$

- **Partition of a sample space :**

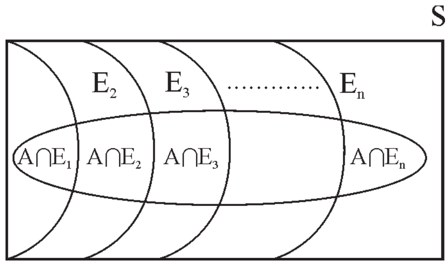
  A set of events $E_1, E_2, \ldots\ldots\ldots, E_n$ is said to represent a partition of a sample space S if

  i) $E_i \cap E_j = \phi$, $i \neq j$; $i, j = 1, 2, \ldots\ldots, n$

  ii) $E_1 \cup E_2 \cup \ldots\ldots\ldots \cup E_n = S$, and

  iii) Each $E_i \neq \phi$, i.e., $P(E_i) > 0$ for all $i = 1, 2, \ldots\ldots, n$

- **Theorem of Total Probability :**

  Let $\{E_1, E_2, \ldots\ldots\ldots, E_n)$ be a partition of the sample space S. Let A be any event associated with S, then

  $$P(A) = \sum_{j=1}^{n} P(E_j) P\left(A/E_j\right)$$

- **Baye's Theorem :**

  If $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive an exhaustive events associated with a sample space, and A is any event of non-zero probability, then

  $$P\left(E_i/A\right) = \frac{P(E_i) P\left(A/E_i\right)}{\sum_{i=1}^{n} P(E_i) P\left(A/E_i\right)}$$

- **Random variable and its probability distribution :**

  A random vairable is a real valued function whose domain is the sample space of a random experiment. If a random variable X takes values $x_1, x_2, \ldots\ldots\ldots, x_n$ with respective pobabilities $p_1, p_2, \ldots\ldots\ldots, p_n$ then

X :	x_1	x_2	....	x_n
P(X) :	p_1	p_2	....	p_n

  is known as the probability distribution of X where, $p_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$.

From the above probability distribution –

$P(X \le x_i) = P(X=x_1) + P(X=x_2) + ... + P(X=x_i)$
$= p_1 + p_2 + ... + p_i$

$P(X < x_i) = P(X=x_1) + P(X=x_2) + ...+ P(X=x_{i-1})$
$= p_1 + p_2 + ... + p_{i-1}$

$P(X \ge x_i) = P(X=x_i) + P(X=x_{i+1}) + ...+ P(X=x_n)$
$= p_i + p_{i+1} + ... + p_n$

$P(X > x_i) = P(X=x_{i+1}) + P(X=x_{i+2}) + ...+ P(X=x_n)$
$= p_{i+1} + p_{i+2} + ... + p_n$

Also,

$P(X \ge x_i) = 1 - P(X < x_i)$, $P(X > x_i) = 1 - P(X \le x_i)$,

$P(X \le x_i) = 1 - P(X > x_i)$ and $P(X < x_i) = 1 - P(X \ge x_i)$

$P(x_i \le X \le x_j) = P(X=x_i) + P(X=x_{i+1}) + ...+ P(X=x_j)$

$P(x_i < X < x_j) = P(X=x_{i+1}) + P(X=x_{i+2}) + ...+ P(X=x_{j-1})$

- **Mean of a discrete random variable :**

If X be a discrete random variable which assumes values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$, then the mean $\overline{X}$ of X is defined as

$$\overline{X} = p_1x_1 + p_2x_2 + ... + p_nx_n$$

or, $$\overline{X} = \sum_{i=1}^{n} p_ix_i$$

- **Remark :**

i) The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by E(X).

ii) In case of a frequency distribution, the mean $\overline{X}$ is given by

$$\overline{X} = \frac{1}{N}\left(f_1x_1 + f_2x_2 + ... + f_nx_n\right)$$

$$\Rightarrow \quad \overline{X} = \frac{f_1}{N}x_1 + \frac{f_2}{N}x_2 + ... + \frac{f_n}{N}x_n$$

$$\Rightarrow \quad \overline{X} = p_1x_1 + p_2x_2 + ... + p_nx_n, \quad \text{where } p_i = \frac{f_i}{N}$$

- **Note :** The mean of a random variable means the mean of its probability distribution.

- **Variance of a discrete random variable :**

  If X is a discrete random variable which assumes values $x_1, x_2, ..., x_n$ with the respective probabilities $p_1, p_2, ..., p_n$, then variance of X is defined as

  $$Var(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2$$

  or, $$Var(X) = E(X^2) - \{E(X)\}^2$$

- **Binomial distribution :**

  A random variable X which takes values 0, 1, 2,........, n is said to follow binomial distribution if its probability distribution function is given by

  $P(X=r) = {}^nc_r p^r q^{n-r}$, r = 0, 1, 2,.........,n where p, q>0 such that p+q=1.

  The two contants n & p in the distribution are known as the parameters of the distribution.

  The notation X~B(n, p) is generally used to denote that the random variable X follows binomial disribution with parameters n and p.

  The probability distritution of the random variable X is given by

X	:	0	1	2	....	r	....	n
P(X)	:	${}^nc_0 p^0 q^{n-0}$	${}^nc_1 p^1 q^{n-1}$	${}^nc_2 p^2 q^{n-2}$	....	${}^nc_r p^r q^{n-r}$	....	${}^nc_n p^n q^{n-n}$

- If n trials constitute an experiment and the experiment is repeated N times, then the frequencies of 0, 1, 2,..........., n successes are given by

  N.P(x = 0), N.P(x = 1), N.P(x = 2),......., N.P(x = n)

- The mean and variance of a binomial variate with parameters n and p are np and npq respectively

- If (n+1)p is not an integer :

  P(X = 0), P(X = 1), ..., P(X = n)

  a) Then, P(X = r) is maximum when r = m = [(n+1)p]

  b) If (n+1)p is an integer, then P(X = r) is maximum when r = m–1 or r = m where m = (n+1)p is an integer.

Exercise- 13
Section-A

OBJECTIVE TYPE QUESTIONS : [1 or 2 marks for each question]

1) Multiple choice type questions :

i) A card is picked at random from a pack of 52 playing cards. Given that the picked card is a King, the probability of this card to be a card of spade is

a) $\frac{1}{3}$ b) $\frac{4}{13}$ c) $\frac{1}{4}$ d) $\frac{1}{2}$

ii) A die is thrown once. Let A be the event that the number obtained is greater than 3 and B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is

a) $\frac{2}{5}$ b) $\frac{3}{5}$ c) 0 d) 1

iii) A number is choosen randomly from numbers 1 to 60. The probability that the chosen number is a multiple of 2 or 5 is

a) $\frac{2}{5}$ b) $\frac{3}{5}$ c) $\frac{7}{10}$ d) $\frac{9}{10}$

iv) From the set {1, 2, 3, 4, 5}, two numbers a and b ($a \neq b$) are choosen at random. The probabilty that $\frac{a}{b}$ is an integer is

a) $\frac{1}{3}$ b) $\frac{1}{4}$ c) $\frac{1}{2}$ d) $\frac{3}{5}$

v) A bag contains 3 white, 4 black and 2 red balls. If 2 balls are drawn at random (without replacement), then the probability that both the balls are white is

a) $\frac{1}{18}$ b) $\frac{1}{36}$ c) $\frac{1}{12}$ d) $\frac{1}{24}$

vi) Three dice are thrown simultaneously. The probability of obtaining a total score of 5 is

a) $\frac{5}{216}$ b) $\frac{1}{6}$ c) $\frac{1}{36}$ d) $\frac{1}{49}$

vii) If one ball is drawn at random from each of three boxes containing 3 white & 1 black, 2 white & 2 black, 1 white & 3 black balls, then the probability that 2 white and 1 black balls will be drawn is

a) $\frac{13}{32}$ b) $\frac{1}{4}$ c) $\frac{1}{32}$ d) $\frac{3}{16}$

viii) A and B draw two cards each, one after another, from a pack of well-shuffled pack of 52 cards. The probability that all the four cards drawn are of the same suit is

a) $\frac{44}{85 \times 49}$ b) $\frac{11}{85 \times 49}$ c) $\frac{13 \times 24}{17 \times 25 \times 49}$ d) none of these

ix) A and B are two events such that P(A)=0.25 and P(B)=0.50. The probability of both happening together is 0.14. The probability of both A and B not happening is

a) 0.39 b) 0.25 c) 0.11 d) none of these

x) The probalities of a student getting I, II and III division in an examination are $\frac{1}{10}, \frac{3}{5}$ and $\frac{1}{4}$ respectively. The probability that the student fails in the examination is

a) $\frac{197}{200}$ b) $\frac{27}{100}$ c) $\frac{83}{100}$ d) none of these

xi) The probability that a leap year will have 53 Fridays or 53 Saturdays is

a) $\frac{2}{7}$ b) $\frac{3}{7}$ c) $\frac{4}{7}$ d) $\frac{1}{7}$

xii) A speaks truth in 75% cases and B speaks truth in 80% cases. Probability that they contradict each other in a statement is

a) $\frac{7}{20}$ b) $\frac{13}{20}$ c) $\frac{3}{5}$ d) $\frac{2}{5}$

xiii) Three integers are choosen at random from the first 20 integers. The probability that their product is even is

a) $\frac{2}{19}$ b) $\frac{3}{29}$ c) $\frac{17}{19}$ d) $\frac{4}{19}$

xiv) A coin is tossed three times. If events A and B are defined as A = Two heads come, B = Last should be head. Then, A and B are

a) independent b) dependent c) both d) none of these

xv) A bag contains 5 brown and 4 white socks. A man pulls out two socks. The probability that these are of the same colour is

a) $\frac{5}{108}$ b) $\frac{18}{108}$ c) $\frac{30}{108}$ d) $\frac{48}{108}$

xvi) If S be the sample space and P(A)=$\frac{1}{3}$ P(B) and S=A∪B, where A and B are two mutually exclusive events, then P(A) =

a) $1/4$ b) $1/2$ c) $3/4$ d) $3/8$

xvii) If A and B are two events, then =

a) $P(\bar{A})P(\bar{B})$ b) 1–P(A)–P(B) c) P(A) + P(B) – P(A∩B) d) P(B) – P(A∩B)

xviii) If P(A∪B)=0.8 and P(A∩B)=0.3, then $P(\overline{A})+P(\overline{B})=$

a) 0.3 b) 0.5 c) 0.7 d) 0.9

xix) The probability that in a year of 22nd century chosen at random, there will be 53 Sunday is

a) $3/28$ b) $2/28$ c) $7/28$ d) $5/28$

xx) From a set of 100 cards numbered 1 to 100, one card is drawn at random. The probability that the number obtained on the card is divisible by 6 or 8 but not by 24 is

a) $6/25$ b) $1/4$ c) $2/5$ d) $4/5$

xxi) A random variable X has the following probability distribution :

X :	1	2	3	4	5	6	7	8
P(X) :	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events E={X is a prime number}, F={X<4}, the probability $P(E \cup F)$ is

a) 0.50 b) 0.77 c) 0.35 d) 0.87

xxii) A random variable X takes the values 0, 1, 2, 3 and its mean is 1.3. If P(X=3)=2P(X=1) and P(X=2)=0.3, then P(X=0) is

a) 0.1 b) 0.2 c) 0.3 d) 0.4

xxiii) A random variable has the following probability distribution :

x :	0	1	2	3	4	5	6	7
p(x) :	0	2p	2p	3p	p^2	$2p^2$	$7p^2$	2p

The value of p is

a) $1/10$ b) –1 c) $-1/10$ d) none of these.

xxiv) If X is a random variable with probability distribution as given below :

X :	0	1	2	3
P(X=x) :	K	3K	3K	K

The value of K and its variance are

a) $1/8, 22/27$ b) $1/8, 23/27$ c) $1/8, 24/27$ d) $1/8, 3/4$

xxv) If X follows a binomial distribution with parameters n = 8 and $p = \frac{1}{2}$, then $P(|X-4| \le 2)$ equals

a) $\frac{118}{128}$ b) $\frac{119}{128}$ c) $\frac{117}{128}$ d) none of these

xxvi) If in a binomial distribution n = 4, $P(X=0) = \frac{16}{81}$, then P(X = 4) equals

a) $\frac{1}{16}$ b) $\frac{1}{81}$ c) $\frac{1}{27}$ d) $\frac{1}{8}$

xxvii) Let X denote the number of times heads occur in n tosses of a fair coin. If P(X = 4), P(X = 5) and P(X = 6) are in A.P; the value of n is/are –

a) 7, 14 b) 10, 14 c) 12, 7 d) 14, 12

xxviii) In a binomial distribution, the probability of getting success is $\frac{1}{4}$ and standard deviation is 3. Then, its mean is

a) 6 b) 8 c) 12 d) 10

xxix) The least number of times a fair coin must be tossed so that the probability of getting at least one head is at least 0.8, is

a) 7 b) 6 c) 5 d) 3

xxx) If X follows a binomial distribution with parameters n =100 and p = $\frac{1}{3}$, then P(x = r) is maximum when r =

a) 32 b) 34 c) 33 d) 31

2] Very short answer type questions :

i) A die is thrown three times, if the first throw is a four, find the chance of getting 15 as the sum.

ii) A coin is tossed three times, if head occurs on first two tosses, find the probability of getting head on third toss.

iii) A coin is tossed, then a die is thrown. Find the probability of obtaining a '6' given that head came up.

iv) If A and B are two events such that $2P(A) = P(B) = \frac{5}{13}$ and $P\left(A/B\right) = \frac{2}{5}$, find $P(A \cup B)$.

v) A die is marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let E be the event 'number is even' and O be the event 'number is red'. Are E & O independent ?

vi) A four digit number is formed using the digits 1, 2, 3, 5 with no repeations. Write the probability that the number is divisible by 5.

vii) 6 boys and 6 girls sit in a row at random. Find the probability that all the girls sit together.

viii) If A and B are two independent events such that P(A)=0.3 and $P\left(A \cup \bar{B}\right)$=0.8. Find P(B).

ix) If A and B are two events then write the expression for the probability of occurance of exactly one of two events.

x) Write the probability that a number selected at random from the set of first 100 natural numbers is a perfect cube.

xi) In a competition A, B and C are participating. The probability that A wins is twice that of B, the probability that B wins is twice that of C. Find the probability that A losses.

xii) If A, B, C are mutually exclusive and exhaustive events associated to a random experiment, then write the value of P(A)+P(B)+P(C).

xiii) If two events A and B are such that $P(\bar{A}) = 0.3$, P(B)=0.4 and $P(A \cap \bar{B}) = 0.5$, find $P\left(B/_{A \cap \bar{B}}\right)$.

xiv) For what value of K the following distribution is a probability distribution ?

$X=x_i$:	0	1	2	3
$P(X=x_i)$:	$2K^4$	$3K^2-5K^3$	$2K-3K^2$	$3K-1$

xv) If X denotes the number on the upper face of a cubical die when it is thrown; find E(X).

xvi) If for a binomial distribution $P(X=1)=P(X=2)=\alpha$, write P(X=4) in terms of α.

xvii) If in a binomial distribution n=4 and $P(X=0)=\frac{16}{81}$, find q.

xviii) If the mean and variance of a binomial variate X are 2 and 1 respectively, find P(X>1).

xix) If the mean of a binomial distribution is 20 and its standard deviation is 4, find p.

xx) Determine the binomial distribution, whose mean is 20 and variance is 16.

Section-B

3] Short answer type questions : [3 marks for each questions]

i) The probability that it will rain on any particular day is 50%. Find the probability that it rains only on first 4 days of the week.

ii) A refrigerator box contains 2 milk chocolates and 4 dark chocolates. Two chocolates are drawn at random. Find the probability distribution of the number of milk chocolates. What is the most likely outcomes.

iii) Given that E and F are events such that P(E)=0.8, P(F)=0.7, P(E∩F)=0.6. Find $P(\bar{E} \mid \bar{F})$.

iv) A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

v) If A and B are two independent events, then prove that the probability of occurance of atleast one of A and B is given by $1-P(A').P(B')$.

vi) If A and B are two events such that P(A)≠0 and $P\left(B/_A\right)=1$, then show that $A \subset B$.

vii) Given two independent events A and B such that P(A) = 0.3 and P(B) = 0.6, find $P(A' \cap B')$.

viii) If P(not A)=0.7, P(B)=0.7 and $P\left(B/_A\right)=0.5$, then find $P\left(A/_B\right)$.

ix) Suppose that 5 men out of 100 and 25 women out of 1000 are good orators. Assuming that there are equal number of men and women, find the probability of choosing a good orator.

x) If A and B are two events such that $A \subset B$ and P(B)≠0, then prove that $P\left(A/_B\right) \geq P(A)$.

xi) Three distinct numbers are chosen randomly from the first 50 natural numbers. Find the probability that all the three numbers are divisible by both 2 and 3.

xii) A speaks truth in 80% cases and B speaks truth in 90% cases. In what percentage of cases are they likely to agree with each other in stating the same fact ?

xiii) If $P(A)=\frac{3}{10}$, $P(B)=\frac{2}{5}$ and $P(A\cup B)=\frac{3}{5}$, then find $P\left(B/A\right)+P\left(A/B\right)$.

xiv) Let $P(A)=\frac{7}{13}$, $P(B)=\frac{9}{13}$ and $P(A\cap B)=\frac{4}{13}$. Then find $P\left(A'/B\right)$.

xv) Let A and B be two given mutually exclusive events. Then find $P\left(A/B\right)$.

xvi) If $P(A)=\frac{2}{5}$, $P(B)=\frac{3}{10}$ and $P(A\cap B)=\frac{1}{5}$, then find $P\left(A'/B'\right)\,P\left(B'/A'\right)$.

xvii) Three balls are drawn from a bag containing 2 red and 5 black balls. If the random variable x represents the number of red balls drawn, then find the value of x and its probability distribution.

Section-C

4. Long answer type questions : [4 or 6 marks]

i) Three balls are drawn one by one with replacement from a bag containing 5 white and 4 red balls. Find the probability distribution of the number of red balls drawn.

ii) The probabilities of A, B and C hitting a target are $\frac{1}{3},\frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three try to shoot the target simultaneously, find the probatility that exactly one of them can shoot it.

iii) A and B take turns in throwing two dice, the first to throw a 9 is awarded the prize. If A throws first then what is the chance that B gets the prize ?

iv) A husband and a wife appear for an interview for two vacancies for the same post. The probability of husband's selection is $\frac{1}{7}$ and that if wife's selection is $\frac{1}{5}$. What is the probability that (a) only one of them will be selected ? (b) at least one of them will be selected ?

v) Bag I contains 4 white and 2 black balls and bag II contains 4 black and 3 white balls. A die is thrown once, if it shows a multiple of 3, bag II is chosen otherwise bag I is chosen and a ball is drawn from the chosen bag. Find the probability that the chosen ball is white ?

vi) For 6 trials of an experiment let x be a binomial variate which satisfies the relation 9P(x=4)=P(x=2). Find P(x=3).

vii) There is a group of 20 persons who are rich, out of these 5 are helpful to poor people. Three persons are selected at random, write the probability distribution for the selected persons who are helpful to poor people. Also find the mean of the distribution.

viii) A coin is tossed once. If it shows a head, it is tossed again but if it shows a tail, then a die is thrown. If all the possible outcomes are equally likely, find the probability that the die shows a number greater than 4, if it is known that the first throw of the coin results in a tail.

ix) There are three coins in which one fair with probability $\frac{1}{2}$ and two biased with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ for a head. One of the coins is tossed twice. If head appears both times, what is the probability that the biased coin with probability $\frac{2}{3}$ for a head is chosen ?

x) If the probabilities that A and B will die with in a year are x and y respectively, find the probability that exactly one of them will be alive at the end of the year.

xi) The probability that a randomly selected voter will vote for party A is 0.2 and the probability that he will vote for party B is 0.5, otherwise he will vote for independent parties. What is the probability that out of 6 voters, 3 or more will vote for party B?

xii) A speaks truth in 75% cases and B speaks truth in 80% cases. In how many cases out of 1000 do you expect them contradict each other in stating the same fact?

xiii) A and B throw alternatively with a pair of dice. A wins if he throws 6 before B and B wins if he throws 7 before A throws 6. If A begins, show that the odds in favour of A are 30:31.

xiv) A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from (a) Calcutta, (b) Tatanagar.

xv) In a class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of the students are boys. If a student is selected at random and found to have an IQ of more than 150, find the probability that the student is a boy.

ANSWERS

Section-A

1)	i) c	ii) d	iii) b	iv) b	v) c	vi) c
	vii) a	viii) a	ix) a	x) b	xi) b	xii) a
	xiii) c	xiv) b	xv) d	xvi) a	xvii) d	xviii) d
	xix) d	xx) a	xxi) b	xxii) d	xxiii) a	xxiv) d
	xxv) b	xxvi) b	xxvii) a	xxviii) c	xxix) d	xxx) c

2)	i) $\frac{1}{18}$	ii) $\frac{1}{2}$	iii) $\frac{1}{6}$	iv) $\frac{11}{26}$	v) No	vi) $\frac{1}{4}$
	vii) $\frac{1}{132}$	viii) $\frac{2}{7}$	ix) $P(A)+P(B)-2P(A\cap B)$		x) $\frac{1}{25}$	xi) $\frac{3}{7}$
	xii) 1	xiii) $\frac{1}{4}$	xiv) $K=\frac{1}{2}$	xv) 3.5	xvi) $\frac{\alpha}{3}$	xvii) $\frac{2}{3}$

xviii) $\frac{15}{16}$ xix) $\frac{1}{5}$ xx) $P(x=r)={}^{100}c_r\left(\frac{1}{5}\right)^r\left(\frac{4}{5}\right)^{100-r}$; $r=0,1,2,....,100$

Section-B

3) i) $\frac{1}{128}$ ii) Most likely outcome is getting one chocolate of each type iii) $\frac{1}{3}$

iv) $\frac{1}{9}$ vii) 0.28 viii) $\frac{3}{14}$ ix) $\frac{3}{80}$ xi) $\frac{1}{350}$ xii) 74%

xiii) $\frac{7}{12}$ xiv) $\frac{5}{9}$ xv) 0 xvi) $\frac{25}{42}$ xvii) x can take values 0, 1, 2

Section-C

4) i)

X	0	1	2	3
P(X)	$\frac{125}{729}$	$\frac{300}{729}$	$\frac{240}{729}$	$\frac{64}{729}$

ii) $\frac{75}{168}$ iii) $\frac{8}{17}$ iv) a) $\frac{2}{7}$ b) $\frac{11}{35}$ v) $\frac{37}{63}$ vi) $540\left(\frac{1}{4}\right)^6$

vii)

X	0	1	2	3
P(X)	$\frac{91}{228}$	$\frac{105}{228}$	$\frac{30}{228}$	$\frac{2}{228}$

, Mean $=\frac{3}{4}$

viii) $\frac{1}{3}$ ix) $\frac{16}{29}$ x) x+y–2xy xi) $\frac{21}{32}$ xii) 350

xiv) a) $\frac{4}{11}$ b) $\frac{7}{11}$ xv) $\frac{3}{7}$

Note

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